

Quantum logic, Hilbert space, revision theory

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Abstract

Our starting point is the observation that with a given Hilbert space H we may, in a way to be made precise, associate a class of non-monotonic consequence relations in such a way that there exists a one-to-one correspondence between the rays of H and these consequence relations. The projectors in Hilbert space may then be viewed as a sort of *revision operators*. The lattice of closed subspaces appears as a natural generalisation of the concept of a Lindenbaum algebra in classical logic. The *logics presentable by Hilbert spaces* are investigated and characterised. Moreover, the individual consequence relations are studied. A key concept in this context is that of a *consequence relation having a pointer to itself*. It is proved that such consequence relations have certain remarkable properties in that they reflect their metatheory at the object level to a surprising extent. The tools used in the investigation stem from two different areas of research, namely from the disciplines of *non-monotonic logic* on the one hand and from *Hilbert space theory* on the other. There exist surprising connections between these two fields of research the investigation of which constitutes the purpose of this paper. © 2001 Published by Elsevier Science B.V.

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1. Introduction

In this paper we establish connections between two seemingly unrelated areas of logical research, namely that of quantum logic on the one hand and that of non-monotonic logic on the other. In particular, we look at the connection between Hilbert space and logic in a new way from the point of view of the theory of non-monotonic consequence relations and belief revision. The (logical) role of Hilbert space is central to quantum logic and to the interpretation of quantum mechanics.

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These areas of research are somewhat heterogeneous in nature and so are the corresponding communities of researchers. There are a variety of logical approaches and an abundance of papers aiming to clarify the foundations of quantum mechanics all of which constitute what has become known as quantum logic. Similarly there are many approaches and subcommunities of research into non-monotonic logic in general and belief revision in particular. It is therefore necessary to say something about what we mean by ‘establishing connections’ between the two fields.

We begin with non-monotonic logic. In non-monotonic logic there are at least two separate lines of investigation. The first line is concerned with attempts at constructing concrete logics and models intended to formalise common sense reasoning. Various sophisticated approaches have been developed for that purpose such as autoepistemic logic, default logic, circumscriptive logic, just to name a few. We may view these investigations as the study of how a non-monotonic consequence relation can be presented in the familiar manner, namely via a formal deductive system.

The second line of investigation, the axiomatic presentation of consequence relations introduced in 1985 by the second author and brought to fruition by the seminal paper by Kraus–Lehmann–Magidor (*KLM*) and many ensuing papers, has two characteristics. The first characteristic is that in this approach non-monotonicity of consequence is studied in a more abstract way via the axiomatic properties of consequence relations without regard to the particular way these consequence relations may be presented. The second characteristic is that, when we actually do present such a consequence relation, then we can also do so semantically, namely by specifying a semantic model for it. Indeed, it is one of the main achievements of this approach to have developed the concept of a model for a consequence relation analogous to the familiar notion of a model for a formula.

Within the area of non-monotonic logic, in the subarea of belief revision, a similar distinction exists. There is the concrete models approach to belief revision, where particular algorithms for revision are put forward, usually making use of specific proof theories. There is also the axiomatic approach, the well known AGM system of postulates for belief revision, which operates on the axiomatic consequence relation level.

In fact, Makinson and Gärdenfors [18] have already observed a connection between general non-monotonic consequence relations ‘ \vdash ’ and revision operations ‘ \circ ’:

- (1) Let Δ be a theory and \circ a revision operator. Define \vdash_{Δ} by

$$A \vdash_{\Delta} B \quad \text{iff} \quad B \in \Delta \circ A.$$

Conversely, assume \vdash is given. Define \circ by

$$\Delta \circ A = \{B \mid \Delta, A \vdash B\}.$$

If we start with a \vdash and define the revision operator \circ associated with it, then \vdash can be retrieved as follows:¹

Let $\Delta_{\emptyset} = \{B \mid \emptyset \vdash B\}$. Then we have

$$\vdash_{\Delta_{\emptyset}} = \vdash$$

¹ We need to assume *theorem monotonicity*: If $\emptyset \vdash C$ then $(A \vdash B \text{ iff } A, C \vdash B)$.

namely:

$$A \vdash_{\Delta\emptyset} B \quad \text{iff} \quad B \in \Delta \circ A \quad \text{iff} \quad B \in \{B \mid \Delta\emptyset, A \vdash B\}.$$

This identity holds for non-monotonic \vdash , satisfying Theorem Monotonicity.

To summarise, we have shown the following:

- (a) Given a \vdash_1 and a revision operator \circ for \vdash_1 and a theory Δ , a new consequence relation \vdash_2 (Δ, \circ, \vdash_1) can be defined by

$$A \vdash_2 B \quad \text{iff} \quad \Delta \circ A \vdash_1 B.$$

- (b) Given a \vdash_2 , a \vdash_1 , \circ and Δ can be found such that $\vdash_2 = \vdash_2(\Delta, \circ, \vdash_1)$.

In fact, we can take \vdash_1 to be \vdash_2 itself, Δ to be $\{X \mid \emptyset \vdash_1 X\}$ and \circ to be defined by $\Gamma \circ A = \{Y \mid \Gamma, A \vdash_1 Y\}$.

- (2) Equipped with the summary (a) and (b) above, let us look at another operation on consequence relations. This operation will play a central role in our paper. Let \vdash be a consequence relation and α a wff. Define a new consequence relation \vdash_α by

$$A \vdash_\alpha B \quad \text{iff} \quad (\text{def}) \quad A, \alpha \vdash B.$$

Consider the mapping

$$\rho_\alpha : \vdash \mapsto \vdash_\alpha.$$

We claim we can view ρ_α as a revision mapping, *revising* \vdash by α to become \vdash_α . How can we show that?

Let us start with \vdash and α . We are going to use (a) above. Let $\vdash_1 = \vdash$ and let $\Delta = \{\alpha\}$. Let \circ be defined as usual by

$$\Gamma \circ A = \{B \mid \Gamma, A \vdash_1 B\}.$$

Consider now $\vdash_2 = \vdash_2(\Delta, \circ, \vdash_1)$.

$$\begin{aligned} A \vdash_2 B & \quad \text{iff} \quad B \in \Delta \circ A \quad \text{iff} \quad \alpha, A \vdash_1 B \\ & \quad \text{iff} \quad \alpha, A \vdash B \quad (\text{since } \vdash_1 = \vdash) \quad \text{iff} \quad A \vdash_\alpha B. \end{aligned}$$

We can therefore view the mapping $(\vdash, \alpha) \mapsto \vdash_\alpha$ as a revision of \vdash by α , analogous to $\Delta \circ A$ being a revision of Δ by A . This view is convenient for establishing the connection between non-monotonic logic and quantum logic. In quantum logic one is naturally led to consider certain classes of consequence relations. Moreover, these classes are closed under certain operators which, in the intuitive sense of the term, should be viewed as revision operators, though not necessarily in the technical sense of standard revision theory.² The reason is as follows. Given a consequence relation \vdash and a formula α inconsistent with \vdash , i.e., we have $\vdash \neg\alpha$. Now, in our situation, the result of revising \vdash by α is the inconsistent (universal) consequence relation. So, in our case, \vdash can only be genuinely revised by α if it is consistent with α . In traditional revision theory, this is the case of expansion. What's the reason for this? Again, note that the consequence relations we are dealing with are so to speak given (by nature) and so is the way they are revised. We are dealing with the way

² We are grateful to the anonymous referee for pointing out that, in the strict sense of traditional revision theory, the term 'expansion operator' would be more appropriate.

revision ‘happens’ in nature as described by quantum mechanics. And ‘in nature’ things are that way. Since this paper differs in motivation from traditional revision theory but on the other hand is concerned with a process that, in the intuitive sense, deserves to be called ‘revision’, we will freely use the term ‘revision’ rather than ‘expansion’. We will come back to this immediately.

What’s the connection between non-monotonic consequence relations and their revision and quantum logic? However we may describe the domain of quantum logic, there is one question which beyond doubt is central to it. This is the question what the mathematical structure of a Hilbert space, which constitutes the core of the formalism of quantum mechanics, has to do with logic. It was first raised by Birkhoff and von Neumann in [2]. We will give a new answer in this paper.

The following intuitive consideration may serve to give an idea of what we have in mind. Imagine a physical system in a pure state x and let A and B denote two observables, say energy and momentum of a particle. Note that an observable need not be ‘sharp’ in state x . This means that as a result of measuring the observable we may get various values each with a certain probability as specified by the statistical formalism of quantum mechanics. We say the observable is sharp with value λ in state x if measurement yields the value λ with certainty. Assume we have a language whose atomic formulas have the intuitive meaning $A = \lambda$, $B = \rho$, ... Suppose that observable A is not sharp in state x . Denote by α the statement $A = \lambda$ and by β the statement $B = \rho$. Assume we measure A getting value λ . According to the projection postulate of quantum mechanics we then end up in a state y in which observable A is sharp, i.e., any subsequent measurement of A yields the value λ . Given that, assume further that in state y observable B is sharp with value ρ . So the above says: “If in state x a measurement of A yields λ , then, after measurement, the system is in a state in which observable B is sharp with value ρ ”. Let us write formally: $\alpha \vdash_x \beta$. This relation \vdash_x (between formulas) turns out to have all the properties a consequence relation is required to have by the general theory of consequence relations and thus deserves to be called a consequence relation. Let us reemphasise that these consequence relations do not arise from any formal deductive system nor do they have anything to do with common sense reasoning. Rather, we may say that they ‘occur’ in nature in the sense that any statement of the form $\alpha \vdash_x \beta$ with α and β being as above is a statement about nature. The reader may remark that the informal statement above should be properly formalised via a connective as a conditional of the form $\alpha \leadsto \beta$. In fact, our approach provides a synthesis of these intuitions via a concept which we call an *internalising connective* the function of which is to reflect the metaconcept of a consequence relation at the object level. Let us, however, point out that the above is by no means a complete description of the nature of the consequence relations we are dealing with. This will become particularly obvious in the last section.

How are these consequence relations presented? The answer is: via Hilbert space! One of the messages of this paper is that the mathematical structure of a Hilbert space, which plays so dominant a role in the quantum mechanical formalism, may be viewed as a vehicle for presenting a system of non-monotonic consequence relations as described above in such a way that there is a one to one correspondence between the consequence relations and the rays of the Hilbert space. In quantum mechanics the rays of a Hilbert space represent the (pure) states of a physical system. So, in the light of the above observation, it seems not unreasonable to regard a physical state, a fundamental notion of quantum mechanics, as a log-

ical entity, namely as a consequence relation. Intuitively, we may think of a physical state as a ‘state of provability’, perhaps more precisely as a ‘state of experimental provability’.

Let us now come back to revision. When and how are these consequences relations revised? First, they are revised in the process of measurement. And, again, it’s via Hilbert space that we learn how the revision is done. Namely, as above, let the physical system be in (pure) state x (in terms of logic: consequence relation \vdash_x). Assume the probability for getting value λ as a result of measuring observable A is non-zero. In terms of logic this means that, denoting the statement $A = \lambda$ by α , we have not $\vdash_x \neg\alpha$. But now assume that we in fact measure λ . Then, physically, this means that according to the projection postulate of quantum mechanics the system is now in a state, say y , such that A is sharp in y , in terms of logic: $\vdash_y \alpha$. There was a transition from state x to state y or, logically, from consequence relation \vdash_x to \vdash_y such that, physically, A is sharp with value λ in y and we have $\vdash_y \alpha$. In order to view this transition as revision, this is one of the most basic conditions revision should satisfy. But let us give another argument in favour of the term ‘revision’. Revision theory requires that among all the consequence relations proving α , \vdash_y should be closest to \vdash_x . In which sense is this the case? It is the case in the sense that y is closest to x in the Hilbert space metric subject to the condition that $\vdash_y \alpha$. In fact, there are different modes how a Hilbert space may serve to present one and the same class of consequence relations. One of these modes is by presenting a model for each of these consequence relations as known from the general theory of (non-monotonic) consequence relations. The notion of closeness plays not only an important role inside these models but also, accounting for revision, between these models.

In order to avoid misunderstandings, let us make clear the difference between traditional revision theory and our approach. Traditional revision theory may be regarded as a theory about the way we humans revise our beliefs and theories in the light of new information. The situation we are confronted with here is different because it is not up to us to revise the consequence relations with which we are dealing. The revision is done by nature and our task consists in studying the properties of this ‘natural’ revision.

Several questions come to mind. What’s the ‘source’ of non-monotonicity in our situation? In the case of commonsense reasoning we may say that it is incomplete information. To answer the question, we will show that there is a close connection between the property of non-monotonicity and a property which we call *self-referential completeness*. Namely, it is a highly remarkable property of the consequence relations we are dealing with that they contain all of their metatheory in a sense that will be made precise. This is a phenomenon unfamiliar from traditional logic and, intuitively, we may view it as the source of non-monotonicity. As to self-referential completeness we suppose that this property is of vital importance for the interpretation of quantum mechanics.

We may ask the question what is the physical correlate of the logical property of non-monotonicity? How is the logical property of non-monotonicity reflected physically? At this stage, we offer a conjecture with which we will deal explicitly in a subsequent paper. The conjecture is that it can be made precise that non-monotonicity is reflected at the physical level as the presence of uncertainty relations. Intuitively speaking, this means that we may view the uncertainty relations as stemming from a ‘logical source’. But, clearly, this needs to be made precise.

But there is another question we can answer precisely in this paper, namely the following. In discussing the quantum mechanical formalism frequently asked questions are: “Why Hilbert space?” or “How does Hilbert space enter quantum mechanics?”. In this paper we cannot discuss whether these questions have ever been precisely posed, let alone answered. We give a precise statement of the problem here and a precise answer. Namely, as already mentioned, Hilbert spaces give rise to systems of consequence relations. We can ask the question: “How can we characterise those systems of consequence relations which are presentable by an (infinite-dimensional) Hilbert space?”. It turns out that the crucial property in that characterisation is a *symmetry property*. So, not only does symmetry play an important role within the quantum mechanical formalism, as is well known from physics, but it is also at the heart of the formalism itself.

The implications our results may have for the interpretation of quantum mechanics remain to be investigated. This is the topic of further work.

2. Plan of the paper

In Section 3 we briefly present the very basics of the general theory of consequence relations. We do so from the syntactic and semantic point of view. We state the minimal syntactic conditions a consequence relation should satisfy. Moreover, we present the concept a model for a consequence relation, which is a slight generalisation of the concept first introduced in the *KLM*-paper as given by the second author. As we will see, these semantic structures ‘occur’ naturally in connection with Hilbert space.

In Section 4 we introduce a concept which we call a *consequence revision system (CRS)*. As already mentioned, the motivation for this comes from Hilbert space and, again, our guide in studying revision in this context is nature itself. We are led to a concept which, for the purposes of this paper, we call a *logic*. Essentially, we define a *logic* to be a *CRS* whose consequence relations share what we call a common *internalising connective*. These concepts arise in a natural way in any logic satisfying the Deduction Theorem. In this sense we may regard our logics as logics satisfying a ‘Generalised Deduction Theorem’. The revision operators naturally form an algebraic structure which we will study. In particular, under certain natural assumptions, these operator algebras are lattices and we can study the interplay of the properties of the consequence relations, which are logical in nature, on the one hand and the properties of the lattice of revision operators, which are algebraic in nature, on the other. The investigation shows that these algebras of revision operators are natural generalisations of the concept of a Lindenbaum algebra as for instance known from classical logic.

In Section 5 we introduce and study the concept of a *Hilbert space logic*. This concept is the link between the two logical disciplines, quantum logic and non-monotonic logic. Essentially, a Hilbert space logic is a logic as explained above which can be presented by a Hilbert space. The individual consequence relations of a Hilbert space logic have remarkable properties. They allow for a sort of self-reference via linguistic devices which we will call *pointers*. For instance, for any formula α the metastatements saying “ α is provable” and “ α is not provable” can be expressed in the object language and, moreover, be proved (by the consequence relation) iff they are true. This fact is the source of what we call self-referential completeness, which means that these consequence relations reflect all

of their metatheory at the object level. It seems to us that this property is of fundamental importance for the interpretation of quantum mechanics, and we will elaborate on this in subsequent work.

In Section 6 we study the semantic structures corresponding to the (syntactic) concepts introduced and studied in Section 4. These structures, which we will call \mathcal{H} -structures, will be systems of models for consequence relations. Again, though motivated by Hilbert space considerations, they occur in a natural way in classical logic too.

In Section 7 we ask the question “Why Hilbert space?” in a precise form. We give a characterisation of those logics that admit to be presented by a classical infinite-dimensional Hilbert space. For this we use an important and relatively recent result of modern Hilbert space theory, namely Solèr’s theorem on the characterisation of classical Hilbert spaces. We introduce the concept of a *symmetric logic* which even from the purely formal point of view is a quite a natural concept in our framework. And, essentially, our main result is that the symmetric logics are exactly those presentable by classical Hilbert spaces. This result highlights a new feature of the role of symmetry with regard to the quantum mechanical formalism. The role of symmetry within that formalism is well known. What our result suggests is that its roots are deeper, namely at the very foundations of the formalism itself.

In Section 8 we take a closer look at the connection between self-referentiality, non-monotonicity and the modal system D .

The references [3–5,8,10,11,16,17] serve to provide the reader with a better understanding of the overall context of the paper.

3. Basics of consequence relations

3.1. Minimal conditions

As already mentioned, we shall be concerned with classes of consequence relations. We shall therefore consider conditions these consequence relations are supposed to satisfy. We always assume that we have a class Fml of formulas closed under the connectives \neg , \wedge , \vee and containing the symbols \top and \perp for truth and falsity respectively. Note that some substitutional instances of truth functional tautologies may not be provable in the consequence relations considered and that we do not require the underlying language to be the language of propositional logic. The only thing we require is that the class of formulas of this language is closed under the above connectives and possibly other connectives. In the case of an infinitary language for instance we would require Fml to be closed under the corresponding infinitary connectives. We consider consequence relations $\vdash \subset Fml \times Fml$ and say that these consequence relations are over Fml .

Consider the following conditions.

<i>Reflexivity</i>	$\alpha \vdash \alpha$
<i>Cut</i>	$\frac{\alpha \wedge \beta \vdash \gamma, \alpha \vdash \beta}{\alpha \vdash \gamma}$
<i>Restricted Monotonicity</i>	$\frac{\alpha \vdash \beta, \alpha \vdash \gamma}{\alpha \wedge \beta \vdash \gamma}$

In the paper by Kraus–Lehmann–Magidor (in short *KLM*, see [15]) these three conditions are, as suggested by Gabbay in [9], considered to be the minimal conditions a consequence relation should satisfy.

As observed in the *KLM* paper, any consequence relation satisfying the above conditions has the following property *AND*:

$$\frac{\alpha \vdash \beta, \alpha \vdash \gamma}{\alpha \vdash \beta \wedge \gamma}.$$

For a given consequence relation \vdash define

$$\alpha \equiv \beta \quad \text{iff} \quad \alpha \vdash \beta \text{ and } \beta \vdash \alpha.$$

Moreover, we impose the following conditions on the consequence relations studied in the present paper:³

$$\begin{aligned} \alpha &\equiv \neg\neg\alpha \\ \top &\equiv \alpha \vee \neg\alpha \\ \perp &\equiv \alpha \wedge \neg\alpha \\ \alpha \wedge \beta &\vdash \alpha \\ \alpha \wedge \beta &\vdash \beta \\ \alpha &\vdash \alpha \vee \beta \\ \beta &\vdash \alpha \vee \beta \\ \vdash \alpha \vee \neg\alpha \\ \alpha &\vdash \top \\ \perp &\vdash \alpha \end{aligned}$$

³ Note that the conditions below are sufficient to prove *Theorem Monotonicity*, namely

$$\frac{\emptyset \vdash C}{A \vdash B \text{ iff } A, C \vdash B} \quad (*)$$

can be proved from these conditions.

To show (*) we need a lemma:

Lemma 1. Assume $A \vdash B$ and $B \vdash A$. Then $C \vdash A$ iff $C \vdash B$.

Proof. From the assumption we get $\neg B \equiv \neg A$. Assume $C \vdash A$ then $\neg A \vdash \neg C$ but $\neg A \vdash \neg B$ hence $\neg A, \neg B \vdash \neg C$ but $\neg B \vdash \neg A$ hence $\neg B \vdash \neg C$ hence $C \vdash B$. \square

We now prove (*):

Assume $\emptyset \vdash C$ and $A \vdash B$. Show $A, C \vdash B$.

Since $\emptyset \vdash C$ and $\emptyset \vdash A \vee \neg A$ we get $A \vee \neg A \vdash C$.

Hence $\neg C \vdash \neg(A \vee \neg A)$. But since $\neg(A \vee \neg A) \equiv \neg A \wedge \neg\neg A$ we get by right equivalence $\neg C \vdash \neg A \wedge \neg\neg A$, hence $\neg C \vdash \neg A$, hence $A \vdash C$. But $A \vdash B$ hence $A, C \vdash B$.

Now assume $A, C \vdash B$ and $\emptyset \vdash C$ and show $A \vdash B$.

Since $\emptyset \vdash \top$ and $\emptyset \vdash C$ we get $\top \vdash C$. But we also have $C \vdash \top$ hence $C \equiv \top$ and by the lemma since $A \vdash \top$ we get $A \vdash C$. Now since $A, C \vdash B$ is given, we get by Cut that $A \vdash B$.

$$\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta.$$

A consequence relation is inconsistent iff for all α, β , $\alpha \sim \beta$, i.e., that \sim is the universal consequence relation denoted by 0 in the sequel. We assume that for the consequence relations we consider this is equivalent to the existence of a formula α such that $\sim \alpha$ and $\sim \neg\alpha$. Any class of consequence relations considered is assumed to contain 0. That means we assume for any $\sim \neq 0$ that for no $\alpha \in Fml$ we have $\sim \alpha$ and $\sim \neg\alpha$. Whenever we speak of a connective in the sequel, we mean a connective definable by $\wedge, \neg, \vee, \top, \perp$. Given a class \mathcal{C} of consequence relations. Then we write $\alpha \sim_{\mathcal{C}} \beta$ iff $\alpha \sim \beta$ for every $\sim \in \mathcal{C}$. We say $\alpha \equiv_{\mathcal{C}} \beta$ iff $\alpha \sim_{\mathcal{C}} \beta$ and $\beta \sim_{\mathcal{C}} \alpha$.

The conditions we imposed so far are local in nature in the sense that they are imposed on every single consequence relation belonging to the class considered. We, moreover, impose the following conditions which have a global character in the sense that they are related to the class \mathcal{C} as a whole.

$$\frac{\alpha \sim_{\mathcal{C}} \gamma, \beta \sim_{\mathcal{C}} \gamma}{\alpha \vee \beta \sim_{\mathcal{C}} \gamma}$$

$$\frac{\alpha \sim_{\mathcal{C}} \beta}{\neg\beta \sim_{\mathcal{C}} \neg\alpha}$$

Definition 1. Let \sim be a consequence relation and \rightsquigarrow a connective. We say that \rightsquigarrow is internalising for \sim iff for all formulas α and β , $\alpha \sim \beta$ is equivalent to $\sim \alpha \rightsquigarrow \beta$. Given a class \mathcal{C} of consequence relations. We say that the connective \rightsquigarrow is an internalising connective for \mathcal{C} iff it is internalising for every $\sim \in \mathcal{C}$.

Note that material implication \rightarrow is internalising for classical consequence. This is a special case of the Deduction Theorem of classical logic.

3.2. On the semantics of consequence relations

The semantics of consequence relations has its origin in investigations on the semantics of conditionals. The essential step in developing semantics of such a sort is the definition of a *model for a consequence relation* rather than a model for a formula. Having accomplished this is one of the merits of the *KLM* paper. In the sequel we give a definition due to Gabbay [6] which is a slight generalisation of the definition given in the *KLM* paper.

Definition 2.

- A Scott model for *Fml* is any function $s : Fml \rightarrow \{0, 1\}$.
- A *GKLM* (Generalised Kraus–Lehmann–Magidor) model is a structure of the form $\langle S, <, l \rangle$, where S is a non-empty set, $<$ is a binary relation on S and l is a function associating with each $t \in S$ a set of Scott models $l(t)$. The model is required to satisfy the smoothness condition stated in the next definition.

Definition 3. Let $\mathcal{M} = \langle S, <, l \rangle$ be a structure as described in the last definition. Let $t \in S$ and α a formula. Then define the satisfaction relation $t \models \alpha$ as follows:

- $t \models \alpha$ iff for all $s \in l(t)$ we have $s(\alpha) = 1$.
- Let $A \subset S$. We say that t is $<$ -minimal in A iff for all $t' \in A$ such that $t' < t$ we have $t' = t$. We say that A is smooth iff for every $t \in A$, either t is minimal in A or for some $s \in A$, $s < t$ and s is minimal in A .
- Let $[\alpha] = \{t \in S \mid t \models \alpha\}$. We say that \mathcal{M} is smooth iff for all α , $[\alpha]$ is smooth.
- For a smooth model \mathcal{M} we define the consequence relation $\vdash_{\mathcal{M}}$ as follows: $\alpha \vdash_{\mathcal{M}} \beta$ iff for all t minimal in $[\alpha]$, we have $t \models \beta$.
- Given a consequence relation \vdash and a smooth model \mathcal{M} . We say \mathcal{M} is a model for \vdash iff $\vdash = \vdash_{\mathcal{M}}$.

4. Formulas as operators

4.1. Classical logic as a motivation

Consider the consequence relation \vdash of classical logic L . As already mentioned, \rightarrow is internalising for \vdash . A trivial but interesting observation is that we can construct a class \mathcal{C}_L of consequence relations in a natural way such that \rightarrow is internalising for all $\vdash \in \mathcal{C}_L$. Given a formula α . Then define \vdash_{α} as follows:

$$\beta \vdash_{\alpha} \gamma \quad \text{iff} \quad \alpha \wedge \beta \vdash \gamma.$$

Since by the Deduction Theorem of classical logic we have $\alpha \wedge \beta \vdash \gamma$ iff $\alpha \vdash (\beta \rightarrow \gamma)$, we see that \rightarrow is internalising for \vdash_{α} .

Define $\mathcal{C}_L = \{\vdash_{\alpha} \mid \alpha \in Fml\}$. It is readily seen that $\vdash_{\alpha} = \vdash_{\beta}$ iff α and β are classically equivalent, i.e., $\alpha \equiv \beta$. Now, given $\vdash_{\alpha} \in \mathcal{C}_L$ and a formula β , then it is natural to look at $\vdash_{\beta \wedge \alpha}$. Again, it is easily verified, using familiar facts of classical logic, that $\vdash_{\beta \wedge \alpha} = \vdash_{\beta' \wedge \alpha}$ iff $\beta \equiv \beta'$. We see that every formula α induces an operator $\bar{\alpha}: \mathcal{C}_L \rightarrow \mathcal{C}_L$ in the way described above. We have the following

- $\bar{\alpha} = \bar{\beta}$ iff $\alpha \equiv \beta$;
- $\bar{\beta}\bar{\alpha} = \bar{\alpha}$ iff $\alpha \vdash \beta$;
- $\bar{\alpha}\vdash_{\beta} = 0$ (inconsistent consequence relation) iff $\vdash_{\beta} \neg\alpha$.

Let us take the above simple observations as a motivation for introducing the concept of an action of a class of formulas on a class of consequence relations, which we shall study in the next subsection.

4.2. Consequence revision systems

Let us now introduce the first key notion of our approach.

Definition 4. Let Fml be a class of formulas as described above and let \mathcal{C} be a class of consequence relations over Fml satisfying the conditions described. Let F be a function $F: Fml \times \mathcal{C} \rightarrow \mathcal{C}$. Then we say that F is an action on \mathcal{C} iff for every consistent $\vdash \in \mathcal{C}$ and $\alpha, \beta \in Fml$ the following conditions are satisfied.

- (i) $F(\top, \sim) = \sim$;
- (ii) $F(\alpha, \sim) = 0$ iff $\sim \neg \alpha$;
- (iii) $F(\beta, F(\alpha, \sim)) = F(\alpha, \sim)$ iff $\alpha \sim \beta$.

If F is an action on \mathcal{C} , we call the pair $\langle \mathcal{C}, F \rangle$ a consequence revision system (CRS).

Note that by $\sim \alpha$ we mean $\top \sim \alpha$. For a given class \mathcal{C} of consequence relations call the formulas α and β \mathcal{C} -equivalent, in symbols $\alpha \equiv_{\mathcal{C}} \beta$, if for every $\sim \in \mathcal{C}$ we have $\alpha \sim \beta$ and $\beta \sim \alpha$.

Remark. In condition (ii) in the above definition it is expressed that from the point of view of general revision theory the action of formulas on consequence relations represents the simple type of revision described in the introduction. So, in the strict sense of traditional revision theory, the term ‘expansion’ would be more appropriate. Since, however, in our most important examples, namely those arising from Hilbert spaces, we are concerned with a process which, in the intuitive sense, deserves to be called revision, we freely use the term ‘revision’. Every $\alpha \in Fml$ induces a (revision) operator on \mathcal{C}

$$\bar{\alpha} : \mathcal{C} \rightarrow \mathcal{C}$$

via

$$\bar{\alpha} \sim =: F(\alpha, \sim).$$

For $\bar{\alpha} \sim$ we will also write \sim_{α} .

Denote the class of these operators by \overline{Fml} . We have $\bar{\alpha} = \bar{\beta}$ iff $\alpha \equiv_{\mathcal{C}} \beta$. \overline{Fml} has the structure of a (multiplicative) semigroup the multiplication being $\bar{\alpha}\bar{\beta} = \bar{\alpha} \circ \bar{\beta}$.

Lemma 2. For any $\alpha \in Fml$ we have $\bar{\alpha}^2 = \bar{\alpha}$.

Proof. By *Reflexivity* we have $\alpha \sim \alpha$ for every $\sim \in \mathcal{C}$. Thus the claim follows by conditions (iii) of the definition of an action. \square

Lemma 3. Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then for any $\sim \in \mathcal{C}$ the following conditions are equivalent

- (i) $\sim \alpha$.
- (ii) $\sim_{\alpha} = \sim$.
- (iii) There exists a $\sim_1 \in \mathcal{C}$ such that $\sim_{1,\alpha} = \sim$.

Proof. For the equivalence of (i) and (ii) observe first that $\sim_{\top,\alpha} = \sim_{\alpha}$. By condition (iii) of the definition of an action we have that $\top \sim \alpha$ iff $\sim_{\alpha} = \sim_{\top,\alpha} = \sim_{\top} = \sim$. Clearly, (ii) implies (iii). In order to show that (iii) implies (ii) suppose $\sim_{1,\alpha} = \sim$. Note that by *Reflexivity* we have $\alpha \sim_1 \alpha$. Then it follows by condition (iii) of the definition of an action that $\sim_{\alpha} = \sim$. \square

Lemma 4. $\alpha \sim \beta$ iff $\sim_{\alpha} \beta$.

Proof. Suppose $\alpha \sim \beta$. By condition (iii) of the definition of an action this is equivalent to $\sim_{\alpha,\beta} = \sim_{\alpha}$. By (i) of the above lemma this means that $\sim_{\alpha} \beta$. \square

It follows by the above two lemmas that $\vdash_\alpha = \vdash_\beta$ implies $\alpha \equiv \beta$, i.e., $\alpha \vdash \beta$ and $\beta \vdash \alpha$. We see that $\overline{\alpha} = \overline{\beta}$ iff $\alpha \equiv_C \beta$.

Definition 5. Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then define the proposition $[\alpha]$ by:

$$[\alpha] = \{\vdash \mid \vdash \alpha\}.$$

We denote the class of propositions of $\langle \mathcal{C}, F \rangle$ by *Prop*.

It is routine to verify the statements made in the following lemma.

Lemma 5. Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then

$$\begin{aligned} \overline{\alpha} \leq \overline{\beta} & \text{ iff } [\alpha] \subset [\beta], \\ \overline{\alpha} = \overline{\beta} & \text{ iff } [\alpha] = [\beta], \\ \alpha \vdash_C \beta & \text{ iff } [\alpha] \subset [\beta], \\ \alpha \equiv_C \beta & \text{ iff } [\alpha] = [\beta], \\ \overline{\alpha} \leq \overline{\beta} & \text{ iff } \overline{\neg\beta} \leq \overline{\neg\alpha}, \\ [\alpha] \subset [\beta] & \text{ iff } [\neg\beta] \subset [\neg\alpha]. \end{aligned}$$

The conditions we imposed on the consequence relations guarantee that the following holds.

Lemma 6. For any CRS both $\langle \overline{Fml}, \leq \rangle$ and $\langle Prop, \subset \rangle$ are lattices. For $\overline{\alpha}, \overline{\beta} \in \overline{Fml}$ and $[\alpha], [\beta] \in Prop$ the greatest lower bounds are $\overline{\alpha} \wedge \overline{\beta}$ and $[\alpha \wedge \beta]$ respectively. The lowest upper bounds are given by $\overline{\alpha} \vee \overline{\beta}$ and $[\alpha \vee \beta]$ respectively.

Given a CRS $\langle \mathcal{C}, F \rangle$. Then define unary operations $*$: $\overline{Fml} \rightarrow \overline{Fml}$ and $*$: $Prop \rightarrow Prop$ as follows:

$$\overline{\alpha}^* = \overline{\neg\alpha} \quad \text{and} \quad [\alpha]^* = [\neg\alpha].$$

Note that in view of Lemma 5 these operations are well defined. Moreover, we define a mapping $\psi: \overline{Fml} \rightarrow Prop$ by

$$\psi(\overline{\alpha}) = [\alpha].$$

By Lemma 5 this mapping is well defined. It is routine to verify the following theorem which displays an analogy to the well known fact that in Hilbert space the lattice of projectors and the lattice of closed subspaces are isomorphic (orthomodular) lattices.

Theorem 1. Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then

- $\langle \overline{Fml}, \leq, * \rangle$ and $\langle Prop, \subset, * \rangle$ are ortholattices.
- ψ is an isomorphism between ortholattices.⁴

⁴ For the definition of an ortholattice or synonymously orthocomplemented lattice see for instance [13] or [20].

Let us reflect a moment at this point. Note that all the concepts introduced so far are metaconcepts. So far there is no concept in our framework that could serve as a link between the metalevel and the object level. The concept of an action of a class of formulas on a class of consequence relations can be viewed as describing connections between consequence relations. But note that all this happens at the metalevel. It is extremely valuable for any logical system to have a concept that can do the job of connecting both levels, the metalevel and the object level. In our framework it is the concept of an *internalising connective* as defined in Section 3.1 that serves this purpose.

Definition 6. Let $\langle \mathcal{C}, F \rangle$ be a CRS and \leadsto a connective. Then we say that \leadsto is an internalising connective for $\langle \mathcal{C}, F \rangle$ iff \leadsto is an internalising connective for all $\vdash \in \mathcal{C}$.

Lemma 7. Let $\langle \mathcal{C}, F \rangle$ be a CRS and let \leadsto be internalising for $\langle \mathcal{C}, F \rangle$. Then the following holds.

- (i) $\alpha \vdash (\beta \leadsto \gamma)$ iff $\beta \vdash_{\alpha} \gamma$.
- (ii) $\{\vdash \mid \alpha \vdash \beta\}$ is a proposition, namely $[\alpha \leadsto \beta]$.

Proof. By Lemma 4 we have $\alpha \vdash (\beta \leadsto \gamma)$ iff $\vdash_{\alpha} (\beta \leadsto \gamma)$. Since \leadsto is internalising, this is equivalent to $\beta \vdash_{\alpha} \gamma$. This proves (i).

(ii) follows from the fact that \leadsto is internalising. \square

Note that in case we have an internalising connective \leadsto the process of revision can be described very simply as follows. Revise the consequence relation \vdash by α so as to get \vdash_{α} . Then γ can be proved from β in \vdash_{α} iff $\beta \leadsto \gamma$ can be proved from α in \vdash .

Given a class of consequence relations \mathcal{C} and two connectives \leadsto_1 and \leadsto_2 . We then say that \leadsto_1 and \leadsto_2 are \mathcal{C} -equivalent iff for all formulas $\alpha, \beta \in Fml$ we have $\alpha \leadsto_1 \beta \equiv_{\mathcal{C}} \alpha \leadsto_2 \beta$.

Lemma 8. Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then any two internalising connectives for $\langle \mathcal{C}, F \rangle$ are \mathcal{C} -equivalent.

Proof. Let \leadsto_1 and \leadsto_2 be two internalising connectives for $\langle \mathcal{C}, F \rangle$. By symmetry it suffices to prove that $\alpha \leadsto_1 \beta \vdash_{\mathcal{C}} \alpha \leadsto_2 \beta$. So let \vdash be any element of \mathcal{C} such that $\vdash \alpha \leadsto_1 \beta$. Since \leadsto_1 is internalising, we have $\alpha \vdash \beta$ and, since \leadsto_2 is internalising, $\vdash \alpha \leadsto_2 \beta$. \square

The above lemma says that the action ‘determines’ the internalising connective modulo \mathcal{C} -equivalence. The next lemma states a sort of converse for this, namely that the internalising connective ‘determines’ the action.

Lemma 9. Let $\langle \mathcal{C}, F_1 \rangle$ and $\langle \mathcal{C}, F_2 \rangle$ be CRS and let \leadsto be a connective which is internalising for both. Then we have $F_1 = F_2$.

Let us now come to a crucial point of our approach. In principle, the concept of an action of formulas on a class of consequence relations can serve a useful purpose. It may

serve as a vehicle for studying the interplay of properties of the operator algebra on the one hand and properties of the class of consequence relations on the other. Generally, properties of the former type are algebraic in nature, whereas properties of the latter type are logical in nature. The orthocomplemented lattice of operators and thus the lattice of propositions may have the algebraic property of being orthomodular and we may ask the question what the logical counterpart of that algebraic property is. The situation we have is, by the way, familiar from various branches of mathematics. It is for instance a familiar technique to study the algebraic structure of certain groups via their action on certain spaces making use of geometrical or topological properties of these spaces. As to the concept of an orthomodular lattice note that this is a dominant concept in virtually all approaches to quantum logic. It is so to speak the quantum logical counterpart of the concept of a Boolean algebra in classical logic. The most prominent examples are the lattices of closed subspaces of a Hilbert space. For the definition of orthomodularity see [13] or [14].

For the proof of the following theorem we use a theorem on orthomodular lattices the proof of which can be found in Mittelstaedt's book [20]. In order to state that result let us, informally, say what we mean by an (ortholattice) conditional. Given an ortholattice L and a function $S: L \times L \rightarrow L$. We say that $S(a, b)$ is an (ortholattice) conditional iff $S(a, b)$ is definable by an 'expression' using the 'constants' a, b and the 'lattice operations' $^*, \wedge, \vee$. For instance $S(a, b) = a^* \vee b$ is such a conditional.

Proposition 1 (Mittelstaedt). *Let $\mathcal{L} = \langle L, \leq, ^* \rangle$ be an ortholattice. Then \mathcal{L} is orthomodular iff there exists an (ortholattice) conditional $S(a, b)$ such that the following conditions are satisfied.*

- (i) $a \wedge S(a, b) \leq b$;
- (ii) $a \wedge c \leq b$ implies $a^* \vee (a \wedge c) \leq S(a, b)$.

A conditional satisfying the above conditions is unique, namely

$$S(a, b) = a^* \vee (a \wedge b).$$

An ortholattice is Boolean iff the above conditions are satisfied by 'material implication', i.e., $S(a, b) = a^ \vee b$.*

Before stating and proving the following theorem let us introduce the following connective \rightsquigarrow_s called *Sasaki hook* defined by:

$$\alpha \rightsquigarrow_s \beta = \neg \alpha \vee (\alpha \wedge \beta).$$

Theorem 2. *Let $\langle \mathcal{C}, F \rangle$ be a CRS such that for any $\vdash \in \mathcal{C}$, $\vdash (\alpha \rightsquigarrow_s \beta)$ implies $\alpha \vdash \beta$ and let \rightsquigarrow be an internalising connective for $\langle \mathcal{C}, F \rangle$. Then $\langle \overline{\text{Fml}}, \leq, ^* \rangle$ and thus $\langle \text{Prop}, \subset, ^* \rangle$ are orthomodular lattices and \rightsquigarrow is \mathcal{C} -equivalent to \rightsquigarrow_s .*

If \rightsquigarrow_s is \mathcal{C} -equivalent to \rightarrow , i.e., material implication, then the above lattices are Boolean algebras.

Proof. In view of Theorem 1 it suffices to prove orthomodularity. We first show that for any $\vdash \in \mathcal{C}$

$$\alpha \wedge (\alpha \rightsquigarrow \beta) \vdash \beta. \tag{1}$$

By Lemma 3 it suffices to show that $\vdash_{\alpha \wedge (\alpha \leadsto \beta)} \beta$. We have, by Lemma 3, $\vdash_{\alpha \wedge (\alpha \leadsto \beta)} \alpha \wedge (\alpha \leadsto \beta)$ and thus $\vdash_{\alpha \wedge (\alpha \leadsto \beta)} \alpha$ and $\vdash_{\alpha \wedge (\alpha \leadsto \beta)} \alpha \leadsto \beta$. Moreover, since \leadsto is internalising, we have $\vdash_{\alpha \wedge (\alpha \leadsto \beta), \alpha} \beta$. But $\vdash_{\alpha \wedge (\alpha \leadsto \beta), \alpha} = \vdash_{\alpha \wedge (\alpha \leadsto \beta)}$, since $\vdash_{\alpha \wedge (\alpha \leadsto \beta)} \alpha$. Now (1) is proved.

It follows that

$$\overline{\alpha} \wedge \overline{\alpha \leadsto \beta} \leq \overline{\beta}. \quad (2)$$

We now prove that the operator $\overline{\alpha \leadsto \beta}$ has the following property.

$$\overline{\alpha} \wedge \overline{\beta} \leq \overline{\gamma} \text{ implies } \overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \gamma}. \quad (3)$$

For this we have to use that every $\vdash \in \mathcal{C}$ satisfies *Cut*. Assume $\overline{\alpha} \wedge \overline{\beta} \leq \overline{\gamma}$ and let $\vdash \in \mathcal{C}$ be such that $\vdash \alpha \leadsto \beta$. We then have $\alpha \wedge \beta \vdash \gamma$ and, since \leadsto is internalising, $\alpha \vdash \beta$. Then we get, using *Cut*, that $\alpha \vdash \gamma$ and again, since \leadsto is internalising, $\vdash (\alpha \leadsto \gamma)$. Thus $\overline{\alpha \leadsto \beta} \leq \overline{\alpha \leadsto \gamma}$.

Now, by the hypothesis, $\vdash (\alpha \leadsto_s \beta)$ implies $\alpha \vdash \beta$ and thus, since \leadsto is internalising, $\vdash (\alpha \leadsto \beta)$. This means $\overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \beta}$. By transitivity we have $\overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \gamma}$.

We have thus proved that, if $\overline{\alpha} \wedge \overline{\beta} \leq \overline{\gamma}$, then $\vdash \alpha \leadsto_s \beta$ implies $\vdash \alpha \leadsto \gamma$ for any $\vdash \in \mathcal{C}$, which means $\overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \gamma}$. We now get by (2), (3) and Proposition 1 that $\langle \overline{Fml}, \leq, * \rangle$ and thus $\langle Prop, \subset, * \rangle$ are orthomodular and, moreover, $\overline{\alpha \leadsto \beta} = \overline{\alpha} * \vee (\overline{\alpha} \wedge \overline{\beta})$. From this it follows that \leadsto and \leadsto_s are \mathcal{C} -equivalent.

That the lattices under consideration are Boolean if \leadsto_s is \mathcal{C} -equivalent to material implication follows by Proposition 1. This completes the proof. \square

Remark. It should be pointed out that in the above proof two properties of the class \mathcal{C} , i.e., logical properties, play a crucial role in establishing the fact that the lattices $\langle \overline{Fml}, \leq, * \rangle$ and $\langle Prop, \subset, * \rangle$ have the algebraic property of being orthomodular. The first (logical) property is that an internalising connective having a certain property exists for $\langle \mathcal{C}, F \rangle$. This property can be viewed as a generalisation of the property that the Deduction Theorem holds. The second crucial property is that all consequence relations of \mathcal{C} satisfy *Cut*.

For the purposes of this paper we introduce the following notion of a *logic*.

Definition 7. Let $\langle \mathcal{C}, F \rangle$ be a CRS and \leadsto an internalising connective for $\langle \mathcal{C}, F \rangle$. Then call the triple $\mathcal{L} = \langle \mathcal{C}, F, \leadsto \rangle$ a logic.

We may thus interpret the above theorem as essentially saying that for a CRS to become a logic (with \leadsto_s as its internalising connective), it is necessary that the lattice of operators $\langle \overline{Fml}, \leq, * \rangle$ and thus the lattice of propositions $\langle Prop, \subset, * \rangle$ have the algebraic property of being orthomodular.

Given a consequence relation \vdash , then define $C(\vdash) =: \{\alpha \mid \vdash \alpha\}$. We have:

Proposition 2. Let $\mathcal{L} = \langle \mathcal{C}, F, \leadsto \rangle$ be a logic. Given $\vdash_1, \vdash_2 \in \mathcal{C}$. Then $C(\vdash_1) = C(\vdash_2)$ iff $\vdash_1 = \vdash_2$.

Proof. Suppose $\mathcal{C}(\vdash_1) = \mathcal{C}(\vdash_2)$ and let $\alpha \vdash_1 \beta$. It follows, since \leadsto is internalising that $\vdash_1 (\alpha \leadsto \beta)$ and thus by the hypothesis $\vdash_2 (\alpha \leadsto \beta)$. Again, since \leadsto is internalising, we get $\alpha \vdash_2 \beta$, thus $\vdash_1 \subset \vdash_2$. By symmetry we also get the other inclusion. \square

4.3. Classical logic revisited

Let us now return to our motivating example from classical logic and look at it from the point of view of the framework developed in the last subsection. Let \vdash denote classical consequence and let $\Sigma \subset Fml$ be any consistent set of formulas. Define the class $\mathcal{C}_{\Sigma, \alpha}$ of consequence relations as follows. For a given formula α define $\vdash_{\Sigma, \alpha}$ by:

$$\beta \vdash_{\Sigma, \alpha} \gamma \quad \text{iff} \quad \Sigma \cup \{\alpha \wedge \beta\} \vdash \gamma.$$

Moreover, define $\mathcal{C}_{L, \Sigma} = \{\vdash_{\Sigma, \alpha} \mid \alpha \in Fml\}$ and the function $\mathcal{F}_{L, \Sigma} : Fml \times \mathcal{C}_{\Sigma, L} \rightarrow \mathcal{C}_{\Sigma, L}$ by $\mathcal{F}_{L, \Sigma}(\beta, \vdash_{\Sigma, \alpha}) = \vdash_{\Sigma, \alpha \wedge \beta}$. It is immediately verified, using familiar facts of classical logic such as the Deduction Theorem, that consequence relations as defined above satisfy all the conditions we imposed and that $\langle \mathcal{C}_{L, \Sigma}, \mathcal{F}_{L, \Sigma} \rangle$ is a CRS.

We have

$$\vdash_{\Sigma, \alpha} = \vdash_{\Sigma, \beta} \quad \text{iff} \quad \Sigma \vdash \alpha \leftrightarrow \beta.$$

Theorem 3. $\mathcal{L}_{L, \Sigma} = \langle \mathcal{C}_{L, \Sigma}, \mathcal{F}_{L, \Sigma}, \rightarrow \rangle$ is a logic. The lattice of operators $\mathcal{O}_{\mathcal{L}_{L, \Sigma}}$ and thus the lattice of propositions $\mathcal{P}_{\mathcal{L}_{L, \Sigma}}$ are Boolean algebras isomorphic to the Lindenbaum algebra $\mathcal{B}(\Sigma)$.

Proof. For the first part of our claim we need to prove that \rightarrow is an internalising connective for $\langle \mathcal{C}_{L, \Sigma}, \mathcal{F}_{L, \Sigma} \rangle$. But this is exactly what the Deduction Theorem says:

$$\Sigma \cup \{\alpha\} \vdash (\beta \rightarrow \gamma) \quad \text{iff} \quad \Sigma \cup \{\alpha \wedge \beta\} \vdash \gamma.$$

It follows from the fact that \rightarrow is internalising and Theorem 2 that the lattices under consideration are Boolean algebras. Moreover, it is straightforward to prove that the following function $\varphi : \mathcal{O}_{\mathcal{L}_{L, \Sigma}} \rightarrow \mathcal{B}(\Sigma)$ is well defined and is an isomorphism

$$\varphi(\overline{\alpha}) = \alpha^\Sigma,$$

where α^Σ denotes the (unique) element of the Lindenbaum algebra $\mathcal{B}(\Sigma)$ to which α belongs. \square

Note that we have established the well known fact that the Lindenbaum algebras of classical logic are Boolean, in a way, however, which is radically different from the method usually applied. If Σ is a complete theory, all the algebras we consider, namely $\mathcal{O}_{\mathcal{L}_{L, \Sigma}}$, $\mathcal{P}_{L, \Sigma}$ and the Lindenbaum algebra $\mathcal{B}(\Sigma)$, are trivial Boolean algebras, i.e., consisting of 0 and 1 only. Since every Boolean algebra is isomorphic to some Lindenbaum algebra, we have the following

Proposition 3. Let \mathcal{B} be any Boolean algebra. Then there exists a class of consequence relations \mathcal{C} and a function F such that $\mathcal{L} = \langle \mathcal{C}, F, \rightarrow \rangle$ is a logic with material implication as its internalising connective and \mathcal{B} is isomorphic to its operator algebra $\mathcal{O}_{\mathcal{L}}$.

The above fact gives rise to the following question:

Is it true that for every orthomodular lattice \mathcal{O} there exists a logic $\mathcal{L} = \langle \mathcal{C}, F, \sim_s \rangle$ with the Sasaki hook as its internalising connective such that \mathcal{O} is isomorphic to $\mathcal{O}_{\mathcal{L}}$?

5. Hilbert space logics

5.1. The concept of a Hilbert space logic

We now study another situation giving rise to a logic of the above type. Let H be a Hilbert space over the real numbers, the complex numbers or the quaternions.⁵ For our purposes we may without loss of generality assume H to be infinite-dimensional. Denote by $Sub(H)$ the class of closed subspaces of H . It is well known that $\langle Sub(H), \subset, \perp \rangle$ is an orthomodular lattice. Note that \perp means orthogonal complement formation. Recall that there exists a one-to-one correspondence between the closed subspaces of H and the projectors of H . We shall use capital letters A, B, \dots for subspaces and, if there is no danger of confusion, for the corresponding projectors. Moreover, we use the symbols for Boolean connectives in connection with closed subspaces, i.e., we write $A \wedge B$ for $A \cap B$ and we denote the smallest closed subspace containing the closed subspaces A and B by $A \vee B$.

Let Fml be a class of formulas closed under \neg, \wedge and containing \top and \perp and let $\Psi : Fml \rightarrow Sub(H)$ be a function such that $\Psi(\alpha \wedge \beta) = \Psi(\alpha) \wedge \Psi(\beta)$ and $\Psi(\neg\alpha) = \Psi(\alpha)^\perp$ and $\Psi(\top) = H$. Denote the projector corresponding to $\Psi(\alpha)$ by A . Let $x \in H$. Then we define the consequence relation \vdash_x by

$$\alpha \vdash_{x,\Psi} \beta \quad \text{iff} \quad Ax \in \Psi(\beta).$$

We shall simply write \vdash_x if Ψ is clear from the context. Note that \vdash_x depends only on the ray of x , i.e., $\vdash_{x_1} = \vdash_{x_2}$ iff the one dimensional subspace $\langle x_1 \rangle$ generated by x_1 is equal to the one dimensional subspace $\langle x_2 \rangle$ generated by x_2 . It is not difficult to prove that these consequence relations in fact satisfy all conditions they are supposed to satisfy, see the proof of the next theorem.

Given a Hilbert Space H and a function Ψ as described above, we define

$$\mathcal{C}_{H,\Psi} = \{\vdash_x \mid x \in H\}.$$

Let us now define a function that will turn out to be an action on $\mathcal{C}_{H,\Psi}$. Define $\mathcal{F}_{H,\Psi} : Fml \times \mathcal{C}_{H,\Psi} \rightarrow \mathcal{C}_{H,\Psi}$ by $\mathcal{F}_{H,\Psi}(\alpha, \vdash_x) = \vdash_{Ax}$. Note that $\mathcal{F}_{H,\Psi}$ is well defined, since $\langle x_1 \rangle = \langle x_2 \rangle$ implies $\langle Ax_1 \rangle = \langle Ax_2 \rangle$.

Theorem 4. *Let H be a Hilbert space, Fml a class of formulas and Ψ a function as described above. Then $\mathcal{L}_{H,\Psi} = \langle \mathcal{C}_{H,\Psi}, \mathcal{F}_{H,\Psi}, \sim_s \rangle$ is a logic.*

⁵ We shall use several results of elementary Hilbert space theory. In the standard textbooks these results are usually proved for Hilbert spaces over the complex numbers. Inspecting the proofs the reader may convince himself that these results equally hold in the real and quaternionic case. In particular, this is for instance easily checked for the Schwarz inequality on which elementary Hilbert space theory rests to a large extent, see for example Rudin's book [23].

We call a logic of the above form a *Hilbert space logic*. Note that in the context of a Hilbert space logic as defined above the elements of the underlying Hilbert space H have a precise meaning, namely as representing consequence relations, which, metaphorically speaking, can be viewed as ‘states of provability’. This is in contrast to the situation we have in quantum mechanics, where the interpretation of the elements of Hilbert spaces as ‘states of a physical system’ is, in our opinion, less clear.

The proof of the above theorem uses well known facts of elementary Hilbert space theory as well as an interesting result by Hardegree, see [12].

Given a Hilbert space H , we denote the norm of an element x by $\|x\|$ and define the metric of H by $d(x, y) = \|x - y\|$.

Proposition 4. *Let H be a Hilbert Space, $x_0 \in H$ and $A \subset H$ a closed subspace of H . Then there exists a unique $x \in A$ such that $d(x, x_0) = \inf_{y \in A} d(y, x_0)$. Moreover, if we denote the projector corresponding to A again by A , we have $x = Ax_0$.*

Proposition 5 (Hardegree). *Let H be a Hilbert space, $A, B \in \text{Sub}(H)$, $x \in H$. Then, if we denote the projectors corresponding to these closed subspaces of H by the same letters, we have $x \in A^\perp \vee (A \wedge B)$ iff $Ax \in B$.*

Given $x \in H$ and $A \in \text{Sub}(H)$, then denote by A_x the unique element of A closest to x .

Proof of Theorem 4. We first need to verify the conditions imposed on the elements of \mathcal{C} . This is routine for the most part. In the following considerations we assume $x \neq 0$.

Reflexivity is a consequence of the fact that $x \in \Psi(\alpha)$ is equivalent to $Ax = x$.

Let us first verify *Cut*. So let $x \in H$ and assume $\alpha \wedge \beta \vdash_x \gamma$ and $\alpha \vdash_x \beta$. It follows that $\Psi(\alpha \wedge \beta)_x = \Psi(\alpha)_x$. $\alpha \wedge \beta \vdash_x \gamma$ says $\Psi(\alpha \wedge \beta)_x \in \Psi(\gamma)$. Thus $\Psi(\alpha)_x \in \Psi(\gamma)$. But this means $\alpha \vdash_x \gamma$. Thus, *Cut* is verified.

Let us verify *Restricted Monotonicity*. Assume $\alpha \vdash_x \beta$ and $\alpha \vdash_x \gamma$. Again, it follows from $\alpha \vdash_x \beta$ that $\Psi(\alpha)_x = \Psi(\alpha \wedge \beta)_x$ and, since by the hypothesis we have $\Psi(\alpha)_x \in \Psi(\gamma)$, we have $\Psi(\alpha \wedge \beta)_x \in \Psi(\gamma)$, which says that $\alpha \wedge \beta \vdash_x \gamma$. Thus *Restricted Monotonicity* is verified.

In order to verify the other conditions, use that by definition we have $\Psi(\alpha \wedge \beta) = \Psi(\alpha) \wedge \Psi(\beta)$ and $\Psi(\neg\alpha) = \Psi(\alpha)^\perp$ and elementary Hilbert space theory.

Verifying the ‘global’ conditions for \mathcal{C} is a straightforward application of elementary Hilbert space theory. For the first global condition for instance suppose $\alpha \vdash_{\mathcal{C}_{H,\Psi}} \gamma$ and $\beta \vdash_{\mathcal{C}_{H,\Psi}} \gamma$. This means $\Psi(\alpha) \subset \Psi(\gamma)$ and $\Psi(\beta) \subset \Psi(\gamma)$. It is then elementary Hilbert space theory that $\Psi(\alpha \vee \beta) \subset \Psi(\gamma)$. This says that $\alpha \vee \beta \vdash_{\mathcal{C}_{H,\Psi}} \gamma$.

We now prove that $\mathcal{F}_{H,\Psi}$ is an action on \mathcal{C} . Condition (i) in Definition 4 is obvious. For condition (ii) in Definition 4 suppose $\vdash_x \neg\alpha$. This is equivalent to $x \in \Psi(\alpha)^\perp$, which is the case iff $Ax = 0$. But this means $\vdash_{Ax} = \mathcal{F}_{H,\Psi}(\alpha, \vdash_x) = 0$. For condition (iii) suppose $\mathcal{F}_{H,\Psi}(\beta, (\mathcal{F}_{H,\Psi}(\alpha, \vdash_x))) = \mathcal{F}_{H,\Psi}(\alpha, \vdash_x)$. This says $\vdash_{BAx} = \vdash_{Ax}$, which is the case iff $BAx = Ax$. But this is equivalent to $Ax \in \Psi(\beta)$, which means $\alpha \vdash_x \beta$.

We still need to prove that \sim_s is internalising for \mathcal{C} . Suppose $\alpha \vdash_x \beta$. By definition this means that $Ax \in \Psi(\beta)$. By Hardegree’s theorem this is the case iff $x \in \neg A \vee (A \wedge B)$, which says $\vdash_x \alpha \sim_s \beta$. \square

5.2. Holicity and the provability of consistency

Certain Hilbert space logics display a property which seems to be worth studying and which we call *holicity*.

Definition 8. Given a Hilbert space logic $\mathcal{L}_{H,\Psi}$. We say that $\mathcal{L}_{H,\Psi}$ is a holistic Hilbert space logic iff for every $x \in H$ there exists a formula σ_x such that $\Psi(\sigma_x) = \langle x \rangle$.

The above definition makes sense, since finite-dimensional subspaces and thus the rays $\langle x \rangle$ are closed. We have the

Lemma 10. Let $\mathcal{L}_{H,\Psi}$ be a holistic Hilbert space logic and $x \in H$ non-zero. Then

- (i) For every x' not orthogonal to x we have $\sigma_x \vdash_{x'} \alpha \rightsquigarrow_s \beta$ iff $\alpha \vdash_x \beta$.
- (ii) $\vdash_x \alpha$ iff $\Psi(\sigma_x \rightsquigarrow_s \alpha) = H$ and thus $\Psi(\neg(\sigma_x \rightsquigarrow_s \alpha)) = \{0\}$.
- (iii) Not $\vdash_x \alpha$ iff $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp$ and thus $\Psi(\neg(\sigma_x \rightsquigarrow_s \alpha)) = \langle x \rangle$.

Proof. (i) We have by elementary Hilbert space theory that $\overline{\sigma_x} \vdash_{x'} = \vdash_x$ if x' is not orthogonal to x , else $\overline{\sigma_x} \vdash_{x'} = 0$. Suppose that x' is not orthogonal to x and $\sigma_x \vdash_{x'} \alpha \rightsquigarrow_s \beta$. By the above remark this is equivalent to $\vdash_x \alpha \rightsquigarrow_s \beta$ and, since \rightsquigarrow_s is internalising, this is the case iff $\alpha \vdash_x \beta$. This proves (i).

(ii) Recall that $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp \vee (\langle x \rangle \wedge \Psi(\alpha))$. $\vdash_x \alpha$ means that $\langle x \rangle \wedge \Psi(\alpha) = \langle x \rangle$. We thus have $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp \vee \langle x \rangle = H$.

(iii) Not $\vdash_x \alpha$ means that $\langle x \rangle \wedge \Psi(\alpha) = \{0\}$ and thus $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp$. \square

We call α x -consistent iff not $\vdash_x \neg\alpha$.

Theorem 5. Let $\mathcal{L}_{H,\Psi}$ be a holistic Hilbert space logic and $x \in H$ non-zero. Then we have

- (i) $\vdash_x \alpha$ iff $\vdash_x \sigma_x \rightsquigarrow_s \alpha$;
- (ii) not $\vdash_x \alpha$ iff $\vdash_x \neg(\sigma_x \rightsquigarrow_s \alpha)$;
- (iii) α is x -consistent iff $\vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$.

Let α be x -consistent. Then we have

- (iv) $\vdash_x \alpha$ iff $\alpha \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$.

Proof. (i) and (ii) follow immediately from Lemma 10. As to (iii) note that the x -consistency of α means that not $\vdash_x \neg\alpha$ and thus (ii) implies (iii).

For (iv) suppose that $\vdash_x \alpha$. This means that $Ax = x$. Since α is x -consistent, we have $\Psi(\neg(\sigma_x \rightsquigarrow_s \neg\alpha)) = \langle x \rangle$ and thus $Ax \in \Psi(\neg(\sigma_x \rightsquigarrow_s \neg\alpha))$, which by definition means $\alpha \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$.

For the other direction assume $\alpha \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$. Since α is x -consistent, we have $Ax \neq 0$. Moreover, since $\neg(\sigma_x \rightsquigarrow_s \neg\alpha) = \langle x \rangle$, $Ax \in \langle x \rangle$. Thus x is an eigenvector of A . The eigenvalue must be 1. So we have $Ax = x$ and thus $\vdash_x \alpha$. \square

Remark. What's the intuitive contents of our notion of holicity? Think, metaphorically, of the consequence relations \vdash_x as 'states of provability' in which certain proofs are possible and others are not. Think of these states of provability as differing in the sense that

certain proofs possible in one state of provability aren't possible in another and vice versa. Moreover, think of the formula σ_x as somehow characterising the state \vdash_x and abbreviate \vdash_x just by x . Let state x be given and assume that β is provable from α in this state, i.e., we have $\alpha \vdash_x \beta$. Now assume we have another state x' not orthogonal to x . Clearly we do not necessarily have $\alpha \vdash_{x'} \beta$, i.e., in state x' it may be the case that we cannot prove β from α . What, however, we can do in state x' is prove that in state x such a proof is possible. We may say that in the case of a holistic Hilbert space logic any state of provability x is encoded in any other state of provability not orthogonal to x via the internalising connective \leadsto_s .

We may interpret the results of the above theorem as follows. Note that in view of (ii) we may say that the formula $\neg(\sigma_x \leadsto_s \alpha)$ expresses 'unprovability' (of α) as $(\sigma_x \leadsto_s \alpha)$ may be viewed as 'provability' (of α). The above theorem says that α is not provable iff the unprovability of α is provable. (iii) says that α is consistent iff its consistency is provable.

Thus, in the case of a holistic Hilbert space logic it is not only the metaconcept of provability that is reflected at the object level but also the metaconcepts of unprovability and consistency. This fact has far-reaching consequences in the sense that the consequence relations of a holistic Hilbert space logic reflect their metatheory at the object level to a surprising extent. We will go into this more deeply in the last section.

(iv) says if α is consistent, it is provable iff it implies its own consistency. A closer look shows that if α is consistent, then it is provable iff it is equivalent to its own consistency.

6. The semantics of consequence revision systems

6.1. The concept of an \mathcal{H} -model

In most of the traditional approaches to logic, a logic can, syntactically, be viewed as a class of formulas and, semantically, the corresponding class of models is a *class of models for formulas*. In our approach, the analogue of the class of formulas in the traditional approaches is given by a *class of consequence relations*. It would, therefore, be natural that the class of models should be a *class of models for consequence relations* rather than a class of models for formulas. For quite some time in the history of logic it was not clear what such a thing could be. As already mentioned, however, such a type of model was put forward in the seminal paper [15] by Kraus–Lehmann–Magidor. We shall use *KLM* as an abbreviation for these three names. The models we use in this paper are *GKLM* (Generalised Kraus–Lehmann–Magidor) models as defined in Section 3.2.

In the last section we saw that, given a Hilbert space H , every $x \in H$ gives rise (via some function Ψ) to a consequence relation $\vdash_{x,\Psi}$ in a natural way. It is a simple but interesting observation that this is not the whole story. Rather, it turns out that not only does x give rise to a consequence relation in a natural way but in an equally natural way also to a *GKLM* model for that consequence relation. Let us make this precise as follows. We always assume $x \neq 0$.

Definition 9. Let H be a Hilbert space, Ψ a function as described in the last section and $x \in H$. Define the binary relation \leq_x on H as follows

$$x_1 \leq_x x_2 \quad \text{iff} \quad d(x, x_1) \leq d(x, x_2).$$

Moreover, define the structure

$$\mathcal{M}_{x,\Psi} = \langle H, \leq_x, l_\Psi \rangle,$$

as follows. Let $x \in H$, then $l_\Psi(x) = \{s_x\}$ is the singleton consisting of the following Scott model s_x : For $\alpha \in Fml$ put $s_x(\alpha) = 1$, if $x \in \Psi(\alpha)$, else $s_x(\alpha) = 0$.

Lemma 11. Let $\mathcal{L}_{H,\Psi}$ be a Hilbert space logic. Then for every $x \in H$, $\mathcal{M}_{x,\Psi}$ is a GKLM model for $\vdash_{x,\Psi}$.

Proof. We first have to verify the smoothness condition. For this observe that for any α we have $[\alpha] = \Psi(\alpha)$. Note that the notation $[\alpha]$ is in the sense of Definition 3 (GKLM). It suffices to show that every $[\alpha]$ has a unique \leq_x -minimal element. But this is what Proposition 5 says, namely Ax is that unique minimal element.

It remains to be shown that $\vdash_{x,\Psi} = \vdash_{\mathcal{M}_{x,\Psi}}$. So let $\alpha \vdash_{x,\Psi} \beta$. By definition this means $Ax \in [\beta]$. This is equivalent to saying that in the GKLM model $\mathcal{M}_{x,\Psi}$ we have $Ax \models \beta$. But this is equivalent to $\alpha \vdash_{\mathcal{M}_{x,\Psi}} \beta$, since Ax is the minimal element of $[\alpha]$. \square

We now propose the following concept of a model for a CRS.

Definition 10. Let $\langle \mathcal{C}, F \rangle$ be a CRS and $\mathcal{H} = \langle H, h, \mathcal{F}, l, g \rangle$ a structure such that

- H is a non-empty set.
- $h : H \rightarrow \mathcal{C}$ is a surjective function.
- $\mathcal{F} : Fml \times H \rightarrow H$ is a function inducing F on $Fml \times \mathcal{C}$ via h , i.e., $F(\alpha, h(x)) = h(\mathcal{F}(\alpha, x))$.
- l is a function assigning to every $x \in H$ a set of Scott models.
- g is an injective function assigning to every $x \in H$ a binary relation $\leq_x \subset H \times H$ such that $\mathcal{M}_x = \langle H, \leq_x, l \rangle$ is a GKLM model for $h(x)$.

Then we say that \mathcal{H} is an \mathcal{H} -model for $\langle \mathcal{C}, F \rangle$. We say that \mathcal{H} is an \mathcal{H} -model for the logic $\langle \mathcal{C}, F, \sim \rangle$ if \mathcal{H} is an \mathcal{H} -model for $\langle \mathcal{C}, F \rangle$. For $x \in H$ and $\alpha \in Fml$ define

$$\langle \mathcal{H}, x \rangle \models \alpha \quad \text{iff} \quad s(\alpha) = 1 \text{ for all } s \in l(x).$$

We say: α is true at x in \mathcal{H} .

Remark. Note that we say that a formula is true at x iff it is provable in $h(x)$.

The following propositions serve to illustrate the nature of \mathcal{H} -models. The proofs are obvious from the definition of an \mathcal{H} -model.

Proposition 6. Let $\langle \mathcal{C}, F \rangle$ be a CRS and \mathcal{H} be an \mathcal{H} -model for $\langle \mathcal{C}, F \rangle$. Let $\sim \in \mathcal{C}$ and $x \in H$ such that $h(x) = \sim$. Then the following conditions are equivalent.

- (i) $\alpha \sim \beta$.
- (ii) $\mathcal{M}_x \models \alpha \sim \beta$.

- (iii) $\mathcal{M}_{\mathcal{F}(\alpha, x)} \models \beta$.
- (iv) $\langle \mathcal{H}, \mathcal{F}(\alpha, x) \rangle \models \beta$.
- (v) $\vdash_{\alpha} \beta$.

Proposition 7. *Let $\mathcal{L} = \langle \mathcal{C}, F, \sim \rangle$ be a logic and \mathcal{H} an \mathcal{H} -model for $\langle \mathcal{C}, F \rangle$. Let $x \in H$. Then the following conditions are equivalent*

- (i) $\alpha \vdash_{\mathcal{M}_x} (\beta \sim \gamma)$.
- (ii) $\langle \mathcal{H}, \mathcal{F}(\alpha, x) \rangle \models \beta \sim \gamma$.
- (iii) $\langle \mathcal{H}, x \rangle \models \alpha \sim (\beta \sim \gamma)$.
- (iv) $\beta \vdash_{\mathcal{M}_{\mathcal{F}(\alpha, x)}} \gamma$.
- (v) $\beta \vdash \gamma$, where $\vdash = F(\alpha, h(x))$.
- (vi) $\alpha \vdash (\beta \sim \gamma)$ with $\vdash = h(x)$.

6.2. The fibred mode of evaluation in \mathcal{H} -models

The notion of an \mathcal{H} -model serves a double purpose. First, it makes sense to speak of the *truth* of a formula in such a model as we are used to from traditional logics and their model theory. Second, these models reflect the following feature of our logics. To see this, recall what intuitively the function of an internalising connective is. An internalising connective serves to reflect the metaconcept of consequence at the object level. So, intuitively, formulas containing the internalising connective ‘talk’ about consequence. \mathcal{H} -models account for this in that they not only model the truth of such formulas but also explicitly model the statements about consequence these formulas make. This means that in the process of evaluation of a formula in an \mathcal{H} -model the internalising connective is evaluated in a *GKLM* model.

Given an \mathcal{H} -model \mathcal{H} , $x \in H$ and a formula of the form $\alpha \sim \beta$. We then have two ways of evaluating the internalising connective \sim . The first way of doing this is to check whether $\langle \mathcal{H}, x \rangle \models (\alpha \sim \beta)$ according to the definition of truth given above. The second way of evaluating the connective \sim is to look at the *GKLM* model \mathcal{M}_x and check whether $\alpha \vdash_{\mathcal{M}_x} \beta$. If so, we have, since \mathcal{M}_x is a *GKLM* model for $h(x) =: \vdash$, $\alpha \vdash \beta$. We have $\alpha \vdash_{\mathcal{M}_x} \beta$ iff $\langle \mathcal{H}, x \rangle \models (\alpha \sim \beta)$. This is how the \mathcal{H} -model reflects the fact that \sim is an internalising connective.

Let us now look at a more complex formula. Consider a formula of the form

$$\varphi = (\alpha \sim (\beta \sim (\gamma \sim \delta))).$$

We may now proceed as follows. We evaluate φ in the *GKLM* model \mathcal{M}_x . We have

$$\begin{aligned} \alpha \vdash_{\mathcal{M}_x} (\beta \sim (\gamma \sim \delta)) \\ \text{iff } \beta \vdash_{\mathcal{M}_{\mathcal{F}(\alpha, x)}} (\gamma \sim \delta) \\ \text{iff } \gamma \vdash_{\mathcal{M}_{\mathcal{F}(\beta, (\mathcal{F}(\alpha, x))}}} \delta. \end{aligned}$$

We have

$$\langle \mathcal{H}, x \rangle \models \varphi \quad \text{iff} \quad \gamma \vdash_{\mathcal{M}_{\mathcal{F}(\beta, (\mathcal{F}(\alpha, x))}}} \delta.$$

The characteristic feature of the second mode of evaluation is that the connective \leadsto is evaluated in *GKLM* models as consequence. At each stage in the process of evaluation we have to switch from one *GKLM* model to another using the ‘fibring function’

$$\mathcal{F}^*: Fml \times \mathcal{M} \rightarrow \mathcal{M}, \text{ where } \mathcal{M} =: \{\mathcal{M}_x \mid x \in H\}$$

defined by

$$\mathcal{F}^*(\alpha, \mathcal{M}_x) = \mathcal{M}_{\mathcal{F}(\alpha, x)}.$$

Note that this ‘fibring function’ is well defined since by the last clause of the definition of an \mathcal{H} -model we have $\mathcal{M}_x = \mathcal{M}_{x'}$ iff $x = x'$. The mode of evaluation just presented is in the spirit of what Gabbay has put forward in several papers and his books [6,7] as fibred semantics.

We will now show that the concept of an \mathcal{H} -model arises in a natural way for instance in connection with the logics $\mathcal{L}_{L, \Sigma} = \langle \mathcal{C}_{L, \Sigma}, \mathcal{F}_{L, \Sigma}, \rightarrow \rangle$, i.e., classical logic, and also with Hilbert space logics.

6.3. \mathcal{H} -models and classical logic

For the notation used in this subsection see Section 4.3.

Definition 11. Let Σ be a set of formulas consistent in classical logic. Consider the structure $\mathcal{H}_{L, \Sigma} =: \langle \mathcal{C}_{L, \Sigma}, h, \mathcal{F}_{L, \Sigma}, l, g \rangle$ such that

- h is the identity function.
- The function l is defined as follows: $l(\vdash_{\Sigma, \alpha}) = \{s_\alpha\}$, where $s_\alpha(\beta) = 1$ if $\vdash_{\Sigma, \alpha} \beta$, else 0.
- The function g is defined as follows. Given $x = \vdash_{\Sigma, \alpha} \in \mathcal{C}_{L, \Sigma}$, then define $g(x) = \leq_\alpha$ as follows: $\vdash_{\Sigma, \beta} \leq_\alpha \vdash_{\Sigma, \gamma}$ is defined only if $\vdash_{\Sigma, \beta} \alpha$. Then, if $\vdash_{\Sigma, \gamma} \alpha$, $\vdash_{\Sigma, \beta} \leq_\alpha \vdash_{\Sigma, \gamma}$ iff $\vdash_{\Sigma, \gamma} \beta$. If not $\vdash_{\Sigma, \gamma} \alpha$, then $\vdash_{\Sigma, \beta} \leq_\alpha \vdash_{\Sigma, \gamma}$.

Note that in the above definition the function l and g are well defined. This is readily seen using familiar facts of classical logic. $\mathcal{C}_{L, \Sigma}$ and $\mathcal{F}_{L, \Sigma}$ are defined in Section 4.3. Note that we use the notation $[\alpha]$ in two different contexts, namely in the context of a *CRS* and in the context of a *GKLM* model. In the present situation the notions coincide, since we have $\vdash_{\Sigma, \alpha} \beta$ iff $s_\alpha(\beta) = 1$.

Lemma 12. If $\vdash_{\Sigma, \beta} \leq_\alpha \vdash_{\Sigma, \gamma}$ and $\vdash_{\Sigma, \gamma} \leq_\alpha \vdash_{\Sigma, \beta}$, then $\vdash_{\Sigma, \beta} = \vdash_{\Sigma, \gamma}$.

Proof. Assume that $\vdash_{\Sigma, \beta} \leq_\alpha \vdash_{\Sigma, \gamma}$ and $\vdash_{\Sigma, \gamma} \leq_\alpha \vdash_{\Sigma, \beta}$. Then we observe, inspecting the definition of \leq_α , that we have both $\vdash_{\Sigma, \beta} \alpha$ and $\vdash_{\Sigma, \gamma} \alpha$. But in this case, again by the definition of \leq_α , the above is only possible if $\vdash_{\Sigma, \beta} \gamma$ and $\vdash_{\Sigma, \gamma} \beta$. From this it follows by well known facts of classical logic that $\vdash_{\Sigma, \beta} = \vdash_{\Sigma, \gamma}$. \square

Lemma 13. $\vdash_{\Sigma, \alpha \wedge \beta}$ is the unique \leq_α -minimal element in $[\beta]$, where $[\beta]$ denotes the proposition represented by β in the logic $\mathcal{L}_{L, \Sigma}$.

Proof. First note that $\vdash_{\Sigma, \alpha \wedge \beta} \in [\beta]$, since $\vdash_{\Sigma, \alpha \wedge \beta} \beta$. Clearly, $\vdash_{\Sigma, \alpha \wedge \beta} \alpha$. Let $\vdash_{\Sigma, \gamma} \in [\beta]$. If not $\vdash_{\Sigma, \gamma} \alpha$ then $\vdash_{\Sigma, \alpha \wedge \beta} \leq_{\alpha} \vdash_{\Sigma, \gamma}$. If $\vdash_{\Sigma, \gamma} \alpha$ then, since $\vdash_{\Sigma, \gamma} \beta$, $\vdash_{\Sigma, \gamma} \alpha \wedge \beta$. But this means $\vdash_{\Sigma, \alpha \wedge \beta} \leq_{\alpha} \vdash_{\Sigma, \gamma}$ for every $\vdash_{\Sigma, \gamma} \in [\beta]$. From this and the last lemma it follows that $\vdash_{\Sigma, \alpha \wedge \beta}$ is a \leq_{α} -minimal element in $[\beta]$. To see that it is unique, let $\vdash_{\Sigma, \delta}$ be any \leq_{α} -minimal element of $[\beta]$. We have $\vdash_{\Sigma, \alpha \wedge \beta} \leq_{\alpha} \vdash_{\Sigma, \delta}$. Since $\vdash_{\Sigma, \delta}$ is \leq_{α} -minimal in $[\beta]$ we get $\vdash_{\Sigma, \delta} = \vdash_{\Sigma, \alpha \wedge \beta}$. \square

Theorem 6. $\mathcal{H}_{L, \Sigma}$ is an \mathcal{H} -model for $\mathcal{L}_{L, \Sigma}$.

Proof. We need to prove that for every $x = \vdash_{\Sigma, \alpha} \in \mathcal{C}_{\Sigma, L}$, $\mathcal{M}_x = \langle \mathcal{C}_{\Sigma, L}, \leq_x, l \rangle$ is a *GKLM* model for x . We have smoothness by Lemma 13. Suppose $\beta \vdash_{\Sigma, \alpha} \gamma$. By definition this means $\Sigma \cup \{\alpha \wedge \beta\} \vdash \gamma$, which is equivalent to $\vdash_{\Sigma, \alpha \wedge \beta} \in [\gamma]$ and our claim follows from Lemma 13 and Definition 3. \square

6.4. The canonical \mathcal{H} -model for a Hilbert space logic

We now define an \mathcal{H} -structure for a given Hilbert space logic $\mathcal{L}_{H, \Psi}$.

Definition 12. Given the Hilbert space logic $\mathcal{L}_{H, \Psi} = \langle \mathcal{C}_{H, \Psi}, \mathcal{F}_{H, \Psi}, \sim_s \rangle$. Consider the structure $\mathcal{H}_{H, \Psi} = \langle H, h, \mathcal{F}, l_{\Psi}, g \rangle$ such that

- $h(x) = \vdash_x$.
- $\mathcal{F}(\alpha, x) = Ax$.
- The function l_{Ψ} is defined as follows: $l_{\Psi}(x) = \{s_x\}$, where $s_x(\alpha) = 1$ if $x \in \Psi(\alpha)$, 0 else.
- $g(x) = \leq_x$ as defined in Definition 9.

Theorem 7. Given a Hilbert space logic $\mathcal{L}_{H, \Psi} = \langle \mathcal{C}_{H, \Psi}, \mathcal{F}_{H, \Psi}, \sim_s \rangle$. Then $\mathcal{H}_{H, \Psi}$ as defined above is an \mathcal{H} -model for $\mathcal{L}_{H, \Psi}$.

7. Hilbert space presentability and symmetry

In the previous sections we investigated some of the properties displayed by Hilbert space logics. In this section we are looking for properties characterising Hilbert space logics. To pose the problem more precisely, let us introduce the following terminology. Given a logic $\mathcal{L} = \langle \mathcal{C}, F, \sim \rangle$, a Hilbert space H and a function $\Psi \rightarrow \text{Sub}(H)$ such that $\mathcal{L} = \mathcal{L}_{H, \Psi}$. Then we say that \mathcal{L} is *presented* by H via Ψ . We say that \mathcal{L} is *presentable* by H if there exists a function Ψ such that \mathcal{L} is presented by H via Ψ . It is our aim to characterise the logics presentable by some Hilbert space H . In other words, we are looking for necessary and sufficient conditions for a logic \mathcal{L} to be presentable by some Hilbert space H . We shall see that, besides some natural logical conditions, there are two properties essential for the characterisation we have in mind. The first property is what in Section 5.2 we called *holicity*. The second essential property, which we haven't come across yet, is a symmetry property by nature. We shall call it the *symmetry property*. We shall see that, essentially, these two properties, namely *holicity* and *symmetry*, characterise Hilbert space

logics. To be more precise, they characterise those logics which are presentable by infinite-dimensional Hilbert spaces, i.e., those structures playing a dominant role in quantum mechanics. Mathematically, the main pillar of our reasoning is Solèr's celebrated result on the characterisation of (infinite-dimensional) Hilbert spaces.

7.1. Some facts from Hilbert space theory

In order to make our considerations as self-contained as possible, we give a short summary of the main theorems of Hilbert space theory we shall use without, however, being fully explicit in our presentation. For details the reader is referred to [13,14,21].

Let K be a (not necessarily commutative) field with an involution τ , i.e., a function $\tau : K \rightarrow K$ such that

$$\tau(a + b) = \tau(a) + \tau(b), \quad \tau(ab) = \tau(b)\tau(a), \quad \tau\tau(a) = a.$$

Now, let H be an infinite-dimensional vector space over K and $\langle \rangle : H \times H \rightarrow K$ be a Hermitian form on H , i.e., $\langle \rangle$ satisfies

$$\begin{aligned} \langle ax + by, z \rangle &= a\langle x, z \rangle + b\langle y, z \rangle, \\ \langle z, ax + by \rangle &= \langle z, x \rangle\tau(a) + \langle z, y \rangle\tau(b), \\ \langle x, z \rangle &= \tau(\langle z, x \rangle). \end{aligned}$$

Call the pair $\langle H, \langle \rangle \rangle$ a *hermitian space*. Call $\langle \rangle$ *anisotropic* iff

$$\langle x, x \rangle = 0 \text{ implies } x = 0.$$

We define the concepts of *orthogonality* of vectors and the orthogonal complement U^\perp of a subspace U as usual. Call a subspace U *closed* iff $U = U^{\perp\perp}$.

Call $\langle H, \langle \rangle \rangle$ an *orthomodular space* iff for every closed subspace U we have

$$H = U \oplus U^\perp.$$

It is well known that the closed subspaces of an orthomodular space H form an atomic, complete, irreducible, orthomodular lattice having the covering property. We denote it by $\text{Sub}(H)$. For the definition of these concepts the reader may for instance consult [13]. However, the reader can follow the proof of Theorem 12, the key result of this section, without being aware of the precise meaning of these terms. We need the following well known key theorem (see for instance [13]).

Theorem 8. *Let L be an atomic, complete, irreducible, orthomodular lattice having the covering property and height at least 4. Then there exists an orthomodular space $\langle H, \langle \rangle \rangle$ such that L is isomorphic to the lattice of subspaces of $\langle H, \langle \rangle \rangle$ in such a way that the atoms of L correspond to the one-dimensional subspaces of $\langle H, \langle \rangle \rangle$.*

We needn't worry about the condition concerning height because the lattices we will be concerned with in the sequel have infinite height.

For quite some time in the history of Hilbert space theory the problem of characterising classical Hilbert spaces, i.e., Hilbert spaces over the real numbers, the complex numbers

or the quaternions, was an open question. It was not before 1995 that the problem was solved by M.P. Solèr (see [26]). The proof was simplified in [22]. The answer to the above question is the following theorem which has become known as Solèr's theorem.

Theorem 9 (Solèr). *Let $\langle H, \langle \rangle \rangle$ be an orthomodular space over K , let there exist an infinite family $(x_i)_{i \in I}$ of pairwise orthogonal elements of H and a non-zero $c \in K$ such that for all $i \in I$ we have $\langle x_i, x_i \rangle = c$. Then K must be the (skew-) field of the real numbers, the complex numbers or the quaternions and H is an infinite-dimensional Hilbert space.*

We call a bijective map $\sigma : H \rightarrow H$ of the orthomodular space H into itself a *unitary operator* on H iff the following conditions are satisfied:

- For any $x, y \in H$ we have $\sigma(x + y) = \sigma(x) + \sigma(y)$.
- There exists an automorphism φ of K such that, for any $\lambda \in K$ and $x \in H$, we have $\sigma(\lambda x) = \varphi(\lambda)\sigma(x)$.
- For any $x, y \in H$, $\langle \sigma(x), \sigma(y) \rangle = \langle x, y \rangle$.

We will use the following result of Mayet (see [19]).

Theorem 10 (Mayet). *Let H be an orthomodular space and let $X \in \text{Sub}(H)$ be of dimension at least 2. Let f be an automorphism of $\text{Sub}(H)$ whose restriction to $[0, X]$ is the identical map. Then there exists a unique unitary operator σ on H inducing f such that the restriction of σ to X is the identical map.*

Note that $[0, X]$ is defined to be $\{G \in \text{Sub}(H) \mid G \subset X\}$ equipped with a lattice structure in a natural way. For the sake of clarity let us point out that by an automorphism we always mean an orthoautomorphism, since we always deal with ortholattices.

7.2. Holistic logics

The key notion introduced in this section is the notion of a (non-trivial) *holistic logic*, which generalises the concept of a holistic Hilbert space logic introduced in Section 5.

Definition 13. Let $\mathcal{L} = \langle \mathcal{C}, F, \sim \rangle$ be a logic. We call \mathcal{L} non-trivial if for every $\alpha \in \text{Fml}$ which is not \mathcal{C} -equivalent to truth or falsity there exists a $\vdash \in \mathcal{C}$ such that neither $\vdash \alpha$ nor $\vdash \neg \alpha$.

Definition 14. Let $\langle \mathcal{C}, F \rangle$ be a CRS, $\vdash_0 \in \mathcal{C}$ a consistent consequence relation and σ a formula not equivalent to falsity. We say that σ is a pointer to \vdash_0 iff for every $\vdash \in \mathcal{C}$ we have $\vdash_\sigma = \vdash_0$ or $\vdash_\sigma = 0$. We also say that \vdash_0 has (σ as) a pointer to itself. We call a non trivial logic \mathcal{L} holistic iff every consistent $\vdash_0 \in \mathcal{C}$ has a pointer to itself.

The following lemma is routinely verified.

Lemma 14. *Let \mathcal{L} be a logic. Then any two pointers are \mathcal{C} -equivalent For a pointer σ to \vdash_0 we have for any $\vdash \in \mathcal{C}$ that $\vdash_\sigma = \vdash_0$ if $\vdash \not\vdash \neg \sigma$, otherwise $\vdash_\sigma = 0$.*

Given two consequence relations \vdash_1 and \vdash_2 with pointers σ_1 and σ_2 respectively. We say that \vdash_1 is *orthogonal* to \vdash_2 iff $\vdash_1 \neg \sigma_2$. This relation is symmetric and we say that the two consequence relations are orthogonal.

The proof of the following lemma is routine.

Lemma 15. *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic, $\vdash_0 \in \mathcal{C}$ having pointer σ to itself. Then*

- (i) *For any $\vdash \in \mathcal{C}$ not orthogonal to \vdash_0 we have $\sigma \vdash (\alpha \rightsquigarrow \beta)$ iff $\alpha \vdash_0 \beta$.*
- (ii) $[\sigma] = \{\vdash_0, 0\}$.
- (iii) $\sigma \vdash_{\mathcal{C}} (\alpha \rightsquigarrow \beta)$ iff $\alpha \vdash_0 \beta$ for any \vdash not orthogonal to \vdash_0 .
- (iv) *For any non-zero $\vdash_1, \vdash_2 \in \mathcal{C}$, $\vdash_1 \subset \vdash_2$ implies $\vdash_1 = \vdash_2$.*

Lemma 16. *Non-trivial holistic Hilbert space logics are holistic.*

Proof. Let $\mathcal{L}_{H,\psi}$ be any non-trivial holistic Hilbert space logic. We need to show that for any non-zero elements x, x' of H we have $\overline{\sigma}_x \vdash_{x'} = \vdash_x$ or $\overline{\sigma}_x \vdash_{x'} = 0$. But this is equivalent to the following fact of elementary Hilbert space theory. Denote by I_x the projection operator corresponding to the ray $\langle x \rangle$. Then $I_x(\langle x' \rangle) = \langle x \rangle$ if x and x' are not orthogonal and $I_x(\langle x' \rangle) = \{0\}$ otherwise. \square

Lemma 17. *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow_s \rangle$ be a holistic logic. Then $\langle \text{Fml}, \leq, * \rangle$ and thus $\langle \text{Prop}, \subset, * \rangle$ are orthomodular, atomic and irreducible lattices.*

Proof. We have orthomodularity by the fact that \mathcal{L} is a logic with the Sasaki hook as its internalising connective and Theorem 2. As to atomicity observe that the atoms of $\langle \text{Prop}, \subset, * \rangle$ are of the form $[\sigma_{\vdash}]$.

For irreducibility we need to prove that the center of that lattice consists of truth and falsity only. For this it suffices to prove that for every proposition $[\alpha]$ not representing truth or falsity there exists an atom $[\sigma_{\vdash}]$ such that $[\alpha]$ and $[\sigma_{\vdash}]$ are not compatible. In the special case of a pointer σ_{\vdash} and a formula α compatibility says that $[\sigma_{\vdash}] \subset [\alpha]$ or $[\sigma_{\vdash}] \subset [\neg\alpha]$. Since \mathcal{L} is nontrivial, for a given formula α there exists a $\vdash_0 \in \mathcal{C}$ such that neither $[\sigma_{\vdash_0}] \subset [\alpha]$ nor $[\sigma_{\vdash_0}] \subset [\neg\alpha]$ and thus $[\alpha]$ and $[\sigma_{\vdash_0}]$ are not compatible. \square

The following theorem gives a connection between local and global consequence in holistic logics. Its proof is routine.

Theorem 11. *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic. Let $\vdash \in \mathcal{C}$. Then the following statements are equivalent.*

- (i) $\vdash \alpha$.
- (ii) $\sigma_{\vdash} \vdash_{\mathcal{C}} \alpha$.
- (iii) $\vdash_{\mathcal{C}} (\sigma_{\vdash} \rightsquigarrow \alpha)$.

Definition 15. Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a logic.

- We say that \mathcal{L} has the upward finiteness property, in brief the uf-property, iff the following holds: Given a set Σ of formulas. Then there exists a formula ψ such that

$\sigma \vdash_{\mathcal{C}} \psi$ for every $\sigma \in \Sigma$ and the following condition is satisfied. For any formula ρ such that $\sigma \vdash_{\mathcal{C}} \rho$ for every $\sigma \in \Sigma$, we have $\psi \vdash_{\mathcal{C}} \rho$.

- We say that \mathcal{L} has the downward finiteness property, in brief the df-property iff the following holds: Given a set Σ of formulas. Then there exists a formula χ such that $\chi \vdash_{\mathcal{C}} \sigma$ for every $\sigma \in \Sigma$ and the following condition is satisfied. For any formula ρ such that $\rho \vdash_{\mathcal{C}} \sigma$ for every $\sigma \in \Sigma$, we have $\rho \vdash_{\mathcal{C}} \chi$.
- In the case that \mathcal{L} is holistic we say that \mathcal{L} has the covering property iff the following condition is satisfied. Given a formula α and $\vdash \in \mathcal{C}$ such that $\not\vdash \alpha$. Then for any formula ρ such that $\alpha \vdash_{\mathcal{C}} \rho$ and $\rho \vdash_{\mathcal{C}} \alpha \vee \sigma_{\vdash}$ we have $\rho \equiv_{\mathcal{C}} \alpha \vee \sigma_{\vdash}$ or $\rho \equiv_{\mathcal{C}} \alpha$.

Intuitively we may think of the formulas ψ and χ in the above definition of playing the role of ‘infinite disjunction’ and ‘infinite conjunction’ of the formulas of Σ . The properties defined above are such that the following lemma holds.

Lemma 18. *Let $\mathcal{L} = \langle \mathcal{C}, F, \sim_s \rangle$ be a holistic logic having the df, uf and the covering properties. Then the lattices $\langle \overline{\text{Fml}}, \leq, * \rangle$ and thus $\langle \text{Prop}, \subset, * \rangle$ are orthomodular, atomic, irreducible, complete lattices having the covering property.*

7.3. Symmetry and Hilbert space logics

Let us start with the following observation. Let H be a Hilbert space and let $(x_i)_{i \in I}$ be a complete orthonormal system of H . Then any permutation of the system $(x_i)_{i \in I}$, more precisely any permutation of the index set I , induces a unique unitary transformation on H and thus an automorphism of the lattice $\text{Sub}(H)$. This fact reflects a symmetry property of Hilbert spaces and in view of Solèr’s theorem seems to be at the heart of the concept of a Hilbert space. It is the above fact that serves us as a motivation for the concept of a symmetric logic which we will study in the sequel.

Definition 16. Let \mathcal{L} be a holistic logic having the properties in the last lemma. Let $\Delta = (\vdash_i)_{i \in I}$ be an infinite family of consequence relations of \mathcal{L} with the following properties:

- For $i \neq j$, \vdash_i and \vdash_j are orthogonal.
- For any consequence relation \vdash of \mathcal{L} there exists an $i_0 \in I$ such that \vdash and \vdash_{i_0} are not orthogonal.

Then we call Δ a basis for \mathcal{L} .

Remark. Intuitively, we may think of a basis Δ of a holistic logic \mathcal{L} as follows. Given any consequence relation \vdash of \mathcal{L} . Then there exists a member of Δ in which \vdash is encoded via the internalising connective. The system Δ may thus be viewed as containing the whole information of \mathcal{L} .

Definition 17. Let \mathcal{L} be a logic as in the last definition and let $\Delta = (\vdash_i)_{i \in I}$ be a basis for \mathcal{L} . We say that \mathcal{L} satisfies the symmetry condition with respect to Δ iff the following holds. Let $f : I \rightarrow I$ be any permutation of the index set I . Then there exists an

automorphism φ_f of the algebra of propositions of \mathcal{L} (and thus of the algebra of operators) such that

- $\varphi_f([\sigma_i]) = [\sigma_{f(i)}]$, where $(\sigma_i)_{i \in I}$ is any family such that σ_i is a pointer to \vdash_i .
- If the subset $J \subset I$ of those elements of I that are left fixed by f is non-empty, then φ_f induces the identity on $[0, A]$, where A is the smallest proposition containing $[\sigma_j]$ for all $j \in J$.

We say that \mathcal{L} satisfies the symmetry condition (synonymously: is symmetric) iff there exists a basis Δ for \mathcal{L} such that \mathcal{L} is symmetric with respect to Δ .

Recall the notation $[0, A]$. It is the set of all propositions smaller than or equal to A equipped with a lattice structure in a natural way. In the following theorem we assume the ‘presenting’ function Ψ to be surjective. We define a classical Hilbert space to be a Hilbert space over the real numbers, the complex numbers or the quaternions.

Theorem 12. *Let $\mathcal{L} = \langle \mathcal{C}, F, \sim_s \rangle$ be a logic. Then the following conditions are equivalent.*

- \mathcal{L} is symmetric.
- There exists an infinite-dimensional classical Hilbert space H presenting \mathcal{L} .

Proof. For the direction from (ii) to (i) assume that there exists an infinite-dimensional classical Hilbert space H and a (surjective) function Ψ such that $\mathcal{L} = \mathcal{L}_{H, \Psi}$. We need to verify the symmetry property. Let $(x_i)_{i \in I}$ be a complete orthonormal system of H . Then $\Delta = (\vdash_{x_i})_{i \in I}$ is a basis for \mathcal{L} . Now observe that the lattice of propositions of \mathcal{L} and $\text{Sub}(H)$ are isomorphic in a canonical way, namely via $[\alpha] \mapsto \Psi(\alpha)$. Thus, for the proof of symmetry it suffices to establish the following. For any permutation $f: I \rightarrow I$ there exists an automorphism ρ_f of $\text{Sub}(H)$ with the following properties:

- $\rho_f(\langle x_i \rangle) = \langle x_{f(i)} \rangle$.
- If the set $J = \{i \mid f(i) = i\}$ is non-empty, then ρ_f induces the identical map on $[0, X]$, where X denotes the smallest closed subspace of H containing $\langle x_j \rangle$ for all $j \in J$.

For this recall that any $x \in H$ has a unique representation of the form $x = \sum_{i \in I} \langle x, x_i \rangle x_i$. Define the map φ_f as follows. For $x = \sum_{i \in I} \langle x, x_i \rangle x_i$ put $\varphi_f(x) = \sum_{i \in I} \langle x, x_{f^{-1}(i)} \rangle x_i$. φ_f is well defined. We have for any $i \in I$ that $\varphi_f(x_i) = x_{f(i)}$. Moreover, φ_f is unitary, since for any $x, y \in H$ we have

$$\langle \varphi_f(x), \varphi_f(y) \rangle = \sum_{i \in I} \langle x, x_{f^{-1}(i)} \rangle \overline{\langle y, x_{f^{-1}(i)} \rangle} = \sum_{i \in I} \langle x, x_i \rangle \overline{\langle y, x_i \rangle} = \langle x, y \rangle.$$

Now assume that the set J of those elements which are left fixed by f is not empty. Denote by X the smallest closed subspace of H containing x_j for all $j \in J$. X is the smallest closed subspace containing $\{\langle x_j \rangle \mid j \in J\}$ and φ_f induces the identity on X . For the latter claim observe that φ_f induces the identity on the subspace spanned by $\{x_j \mid j \in J\}$ and X is the (topological) closure of that subspace. By continuity φ_f induces the identity on X too. Now, φ_f induces an ortholattice automorphism ρ_f on $\text{Sub}(H)$ such that for any $i \in I$, $\rho_f(\langle x_i \rangle) = \langle x_{f(i)} \rangle$. It is also evident that ρ_f induces the identical map on $[0, X]$. Thus the symmetry condition is verified.

For the other direction note that the existence of a basis guarantees that the lattice of propositions denoted by $Prop_{\mathcal{L}}$ has infinite height and observe that by Lemma 18 and Theorem 8 there exists an orthomodular space H and an isomorphism $\Phi : Prop_{\mathcal{L}} \rightarrow Sub(H)$. We now exploit the symmetry property of \mathcal{L} to prove that H must be a classical (infinite-dimensional) Hilbert space. Let Δ be a basis for \mathcal{L} with respect to which \mathcal{L} is symmetric. Let $(\sigma_i)_{i \in I}$ be a corresponding family of pointers. We look at the family $\Phi([\sigma_i])_{i \in I}$. Put $\langle x_i \rangle = \Phi([\sigma_i])$. This is an infinite pairwise orthogonal family of one dimensional subspaces (rays) of H . We will construct a family $(y_i)_{i \in I}$ of pairwise orthogonal elements of H such that for any $i, j \in I$ we have $\langle y_i, y_i \rangle = \langle y_j, y_j \rangle$. Then it follows by Soler's theorem that H is a classical Hilbert space.

Let $i_0 \in I$ be fixed. Then for every $j \in I, j \neq i_0$ consider the permutation f_j of I defined as follows

$$f_j(i_0) = j \text{ and } f_j(j) = i_0, \quad f_j(i) = i \text{ else.}$$

The symmetry condition then guarantees that for every $j \in I, j \neq i_0$ there exists an automorphism φ_j of $Sub(H)$ such that

$$\varphi_j(\langle x_{i_0} \rangle) = \langle x_j \rangle$$

and, moreover, induces the identity on $[0, X]$, where X is the smallest closed subspace of H containing $\langle x_i \rangle$ for $i \neq i_0, j$. Clearly X has dimension greater than 2. In fact, it is infinite-dimensional. Mayet's theorem then yields that φ_j is induced by some unitary operator ρ_j of H . For $j \neq i_0$ put $y_j = \rho_j(x_{i_0})$ and $y_{i_0} = x_{i_0}$. Since ρ_j is unitary and the $\langle x_i \rangle$'s are pairwise orthogonal, the family $(y_j)_{j \in I}$ is as required in Soler's theorem and H must be an infinite-dimensional classical Hilbert space.

We still need to prove that H presents \mathcal{L} . For this we first need to define the function Ψ . Define $\Psi : Fml \rightarrow Sub(H)$ by $\Psi(\alpha) = \Phi([\alpha])$. It is routinely verified that Ψ satisfies the conditions required.

We need to show

- (1) $\mathcal{C} = \mathcal{C}_{H, \Psi}$.
- (2) If $\sim = \vdash_x$, then for any $\alpha, \vdash_\alpha = \mathcal{F}_{H, \Psi}(\alpha, \vdash_x)$.

For (1) let $\sim \in \mathcal{C}$ be given. We need to find a $\vdash_x \in \mathcal{C}_{H, \Psi}$ such that $\sim = \vdash_x$. Let σ be a pointer to \sim and $\langle x \rangle = \Phi([\sigma]) = \Psi(\sigma)$. We have $\alpha \sim \beta$ iff $\sigma \vdash \alpha \rightsquigarrow_s \beta$ iff $[\sigma] \subset [\alpha \rightsquigarrow_s \beta]$. This is equivalent to $\langle x \rangle \subset \Psi(\alpha \rightsquigarrow_s \beta)$ which says $\alpha \vdash_x \beta$. Thus $\sim = \vdash_x$. For a given \vdash_x the same reasoning applies to find a $\sim \in \mathcal{C}$ such that $\vdash_x = \sim$.

For (2) let $\sim = \vdash_x$. Note that $\beta \vdash_\alpha \gamma$ iff $\alpha \sim (\beta \rightsquigarrow_s \gamma)$ iff $\alpha \vdash_x (\beta \rightsquigarrow_s \gamma)$ iff $\beta \vdash_{Ax} \gamma$. But $\vdash_{Ax} = \mathcal{F}_{H, \Psi}(\alpha, \vdash_x)$. \square

Remark. The reader may have noticed that in the above proof we did not use the second condition of the definition of a basis. In fact the argument works without that condition. If we omit the second condition we can no longer say that Δ contains 'the whole information' of \mathcal{L} . Instead, its intuitive function would be to guarantee that \mathcal{L} is 'rich in information' in

that it contains infinitely many non orthogonal consequence relations, which thus are not encoded in each other.⁶

8. Self-referentiality in Hilbert space logics

8.1. Pointing, self-referentiality and non-monotonicity

In this section we introduce the notion of self-referential completeness in connection with consequence relations. This notion was first introduced by Smullyan in [24,25] in connection with modal systems. The results of this section hold for holistic logics. We shall prove that a consequence relation having a pointer to itself is self-referentially complete and, with a certain trivial exception, non-monotonic.

We now define a metalanguage in which we can talk about provability. Intuitively, $DER(\alpha, \beta)$ means ‘ β is derivable from α in \vdash ’.

Definition 18.

- If α, β are wffs of the object language, then $DER(\alpha, \beta) \in ML$.
- If α is a wff of the object language and $\varphi \in ML$, then $DER(\alpha, \varphi) \in ML$ and $DER(\varphi, \alpha) \in ML$.
- If $\varphi, \psi \in ML$, then $DER(\varphi, \psi) \in ML$.
- If $\varphi, \psi \in ML$, so are $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi$.

We use the following abbreviations:

$$PROV\alpha =: DER(\top, \alpha),$$

$$CON\alpha =: \neg PROV\neg\alpha,$$

$$EQUIV(\alpha, \beta) =: DER(\alpha, \beta) \wedge DER(\beta, \alpha).$$

We now define a natural translation of the metalanguage ML into the object language. We assume that we have a logic $\mathcal{L} = \langle \mathcal{C}, F, \sim \rangle$. The following definitions are relative to a fixed $\vdash \in \mathcal{C}$ having a pointer σ to itself.

⁶ The authors cannot refrain from putting themselves in a mystic’s boots for a moment. What’s the intuitive content of the combination of holicity and symmetry which obviously is at the heart of the logics presentable by infinite-dimensional Hilbert spaces? A mystic might say that these properties represent the inherent ‘unity’ of a Hilbert space logic. Given a holistic logic \mathcal{L} , let Δ be a basis for \mathcal{L} and let \mathcal{L} be symmetric with respect to Δ . Then we know that every consequence relation \vdash_0 of \mathcal{L} is encoded in some member \vdash_1 of Δ via the internalising connective and vice versa. So, \vdash_0 and \vdash_1 aren’t essentially different. This is holicity. But what about different elements $\vdash_0, \vdash_1 \in \Delta$? Aren’t they essentially different? Well, the mystic might say, symmetry expresses a sort of indistinguishability of the basic consequence relations, the sort of indistinguishability we encounter so often at the level of quantum mechanics, for instance in connection with the symmetry of the wave function of many particle systems. This time, the symmetry is at the logical level. So, the mystic might say, there is some ‘hidden unity’ behind the apparent diversity and variety of the consequence relations of a Hilbert space logic.

Definition 19. Let σ be a pointer to \vdash . Define the translation $'$ as follows.

- (i) If $\varphi = DER(\alpha, \beta)$ where α and β are wff of the object language, $\varphi' = \sigma \rightsquigarrow (\alpha \rightsquigarrow \beta)$.
- (ii) If $\varphi = DER(\alpha, \psi)$, where α is a wff of the object language and $\psi \in ML$, then $\varphi' = \sigma \rightsquigarrow (\alpha \rightsquigarrow \psi')$; analogously for the case $DER(\psi, \alpha)$.
- (iii) If $\varphi = DER(\psi, \rho)$, where $\psi, \rho \in ML$, $\varphi' = \sigma \rightsquigarrow (\psi' \rightsquigarrow \rho')$.
- (iv) If $\varphi = \neg\psi$, $\varphi' = \neg(\sigma \rightsquigarrow \psi')$.
- (v) If $\varphi = \psi \wedge \rho$, $\varphi' = \psi' \wedge \rho'$; analogously for the other connectives.

We now define the notion of truth for ML in a natural way. This definition of truth is in the spirit of what Smullyan calls a self-referential interpretation (see his book “Gödel’s Incompleteness Theorems” [25] and “Forever Undecided” [24]). Just recall the essential feature of the definition of self-referential truth. Given a modal system M . Then we say that a formula of the form $\Box A$ is (self-referentially) true with respect to M iff A is provable in M .

Definition 20.

- (i) If $\varphi = DER(\alpha, \beta)$, where α, β are wffs of the object language, then $TRUE \varphi$ iff $\alpha \vdash \beta$.
- (ii) If $\varphi = DER(\alpha, \psi)$, where α is a wff of the object language, then $TRUE \varphi$ iff $\alpha \vdash \psi'$; analogously for the case $DER(\psi, \alpha)$.
- (iii) If $\varphi = DER(\psi, \rho)$ for $\psi, \rho \in ML$, then $TRUE \varphi$ iff $\psi' \vdash \rho'$.
- (iv) If $\varphi = \neg\psi$, then $TRUE \varphi$ iff not $TRUE \psi$; analogously for the other connectives.

Definition 21. Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a logic, $\vdash \in \mathcal{C}$ and σ a pointer to \vdash . We say that \vdash is self-referentially complete iff for all $\varphi \in ML$ we have

$$TRUE \varphi \quad \text{iff} \quad \vdash \varphi'.$$

Lemma 19. Given consistent $\vdash_0, \vdash \in \mathcal{C}$, let σ be a pointer to \vdash_0 such that $\not\vdash \neg\sigma$. Then

- (i) $\alpha \vdash_0 \beta$ iff $\vdash \sigma \rightsquigarrow (\alpha \rightsquigarrow \beta)$.

Moreover, we have

- (ii) $\vdash \alpha$ iff $[\sigma \rightsquigarrow \alpha] = [\top]$ and thus $[\neg(\sigma \rightsquigarrow \alpha)] = [\perp]$.
- (iii) $\not\vdash \alpha$ iff $[\sigma \rightsquigarrow \alpha] = [\neg\sigma]$ and thus $[\neg(\sigma \rightsquigarrow \alpha)] = [\sigma]$.

Proof. For (i) recall that $\vdash_\sigma = \vdash_0$. The claim then follows by the fact that \rightsquigarrow is internalising and Lemma 4.

(ii) For the direction from left to right suppose $\vdash \alpha$ and note that for any \vdash_1 such that $\vdash_1 \neg\sigma$ we have $\vdash_1 \sigma \rightsquigarrow \alpha$. If $\not\vdash_1 \neg\sigma$, we have by (i) that $\vdash \sigma \rightsquigarrow \alpha$. Thus $[\sigma \rightsquigarrow \alpha] = \mathcal{C} = [\top]$. The other direction is obvious.

(iii) Suppose that $\not\vdash \alpha$. Then, again, we have for every \vdash_1 such that $\vdash_1 \neg\sigma$, $\vdash_1 \sigma \rightsquigarrow \alpha$. But if $\not\vdash_1 \neg\sigma$, $\vdash_1 \sigma \rightsquigarrow \alpha$ cannot hold, since this would imply $\vdash \alpha$ contrary to the hypothesis. Thus $[\sigma \rightsquigarrow \alpha] = [\neg\sigma]$. The other direction is obvious. \square

Remark. We may view the formula $\neg(\sigma \rightsquigarrow \alpha)$ as expressing the unprovability of α at the object level. It thus follows from (iii) of the above lemma that, if α is not provable,

then its unprovability can be proved. This is a remarkable fact unfamiliar from classical logic which has far-reaching consequences concerning self-referential completeness and non-monotonicity.

Theorem 13. *Let the consequence relation \vdash have a pointer σ to itself. Then \vdash is self-referentially complete.*

Proof. By induction on the construction of the formulas of *ML*.

(i) Case $\varphi = \text{DER}(\alpha, \beta)$. By definition *TRUE* φ means $\alpha \vdash \beta$. But this means $\vdash \alpha \leadsto \beta$, which is equivalent to $\vdash \sigma \leadsto (\alpha \leadsto \beta)$. But this says that $\vdash \varphi'$.

(ii) Case $\varphi = \text{DER}(\alpha, \psi)$. Suppose *TRUE* φ . By definition this says $\alpha \vdash \psi'$ or equivalently $\vdash \sigma \leadsto (\alpha \leadsto \psi')$. But this is exactly what $\vdash \varphi'$ means.

(iii) Case $\varphi = \text{DER}(\psi, \rho)$. The proof is analogous to (ii).

(iv) Case $\varphi = \neg\psi$. *TRUE* φ means that not *TRUE* ψ . By the induction hypothesis this is equivalent to not $\vdash \psi'$, which by ‘provability of unprovability’ says that $\vdash \neg(\sigma \leadsto \psi')$. But this means $\vdash \varphi'$.

(v) Case $\varphi = \psi \vee \rho$. First note that $\varphi' = \psi' \vee \rho'$. Suppose *TRUE* φ . It follows that *TRUE* ψ or *TRUE* ρ . Without loss of generality assume *TRUE* ψ . By the induction hypothesis we have $\vdash \psi'$ and thus $\vdash \psi' \vee \rho'$. But this says $\vdash \varphi'$.

For the other direction suppose $\vdash \varphi'$. We need to prove that *TRUE* φ . There is a problem here, namely that generally, i.e., for arbitrary ψ and ρ , $\vdash \psi \vee \rho$ does not imply that $\vdash \psi$ or $\vdash \rho$. To overcome this obstacle we observe by inspecting the definition of the translation that any formula occurring as a translation is of the form $\sigma \leadsto \dots$ or $\neg(\sigma \leadsto \dots)$ or a Boolean combination of such formulas. It then follows by Lemma 19 that the propositions $[\psi']$ and $[\rho']$ are of the form $[\top]$, $[\perp]$, $[\sigma]$, $[\neg\sigma]$. We can thus treat this case by checking all combinations.

Suppose for instance that $[\psi'] = [\top]$ and $[\rho'] = [\neg\sigma]$. Then $\vdash \psi' \vee \rho'$ says $\vdash \top \vee \neg\sigma$, which is equivalent to $\vdash \top$, i.e., $\vdash \psi'$. It follows by the induction hypothesis that *TRUE* ψ and thus *TRUE* φ .

The other combinations can be checked in an analogous manner. The same applies to the other cases. \square

Remark. Inspecting the translation of the metalanguage into the object language, we may view the metalanguage as a ‘sublanguage’ of the object language. The peculiar feature of this ‘sublanguage’ is that it contains a ‘proof operator’, namely ‘ $\sigma \leadsto$ ’, as opposed to ‘proof predicates’ which we have in other languages. Our notion of self-referentiality thus becomes fully analogous to that introduced by Smullyan in connection with self-application of modal systems, where the modal operator \Box plays the role of a ‘proof operator’. Another way of proceeding would have been this. We could have introduced the metalanguage as a sublanguage of the object language from the outset thus avoiding the need for a translation. In particular, we could have defined self-referential truth without recourse to the translation.

Example. Let us consider an example and let us for the sake of illustration verify the claim made in the above theorem directly in this particular case. Let α be an object formula and consider the following metastatement

$$\varphi = \text{PROV}\alpha \rightarrow \text{CON}\alpha.$$

Its translation is

$$\varphi' = (\sigma \rightsquigarrow (\top \rightsquigarrow \alpha)) \rightarrow \neg(\sigma \rightsquigarrow (\sigma \rightsquigarrow (\top \rightsquigarrow \neg\alpha))).$$

Let us first verify that $\text{TRUE } \varphi$ implies $\vdash \varphi'$. Assume that not $\text{TRUE } \text{PROV}\alpha$. This means $\text{TRUE } \neg \text{PROV}\alpha$, which says that $\nmid \alpha$. By Lemma 19 we have $[\neg(\sigma \rightsquigarrow (\top \rightsquigarrow \alpha))] = [\sigma]$. Thus $[\varphi'] = [\sigma \vee \dots]$ and we have $\vdash \varphi'$.

Now assume $\text{TRUE } \text{CON}\alpha$, i.e., $\nmid \neg\alpha$ and thus $\nmid \top \rightsquigarrow \neg\alpha$, hence $\nmid \sigma \rightsquigarrow (\top \rightsquigarrow \neg\alpha)$. In this case we have by Lemma 19 $[\neg(\sigma \rightsquigarrow (\sigma \rightsquigarrow (\top \rightsquigarrow \neg\alpha)))] = [\sigma]$. Thus $[\varphi'] = [\dots \vee \sigma]$ and we have $\vdash \varphi'$.

Let us now verify that $\vdash \varphi'$ implies $\text{TRUE } \varphi$. So assume $\vdash \varphi'$. By Lemma 19 $[\neg(\sigma \rightsquigarrow (\sigma \rightsquigarrow (\top \rightsquigarrow \alpha)))]$ equals either $[\perp]$ or $[\sigma]$. In the first case we have $\vdash \alpha$. Since \vdash is assumed to be consistent, we have $\nmid \neg\alpha$, which means $\text{TRUE } \text{CON}\alpha$. But this says that $\text{TRUE } \varphi$. In the second case we have $\nmid \alpha$ and thus not $\text{TRUE } \text{PROV}\alpha$ in which case again $\text{TRUE } \varphi$.

Theorem 14. Let \vdash have a pointer to itself. Then \vdash does not admit Gödelian fixed points, i.e., formulas equivalent to their own unprovability. This means that there exists no formula such that

$$(i) \quad \alpha \vdash \neg(\sigma \rightsquigarrow \alpha)$$

and

$$(ii) \quad \neg(\sigma \rightsquigarrow \alpha) \vdash \alpha.$$

Moreover, if α is consistent, (i) does not hold.

Corollary 1. If there exists a formula α such that neither $\vdash \alpha$ nor $\vdash \neg\alpha$, then \vdash is non-monotonic.

If we call a consequence relations *complete* if it has the above property, the above corollary thus says that a consequence relation admitting a pointer to itself is non-monotonic unless it is complete. We shall see shortly that the only ‘monotonic case’ is the case of a complete classical theory.

Proof. Assume there exists a Gödelian fixed point α . Suppose $\vdash \alpha$. Then it follows from (i) that $\vdash \neg(\sigma \rightsquigarrow \alpha)$. By Lemma 19 we have $\nmid \alpha$, contrary to the hypothesis. Now suppose $\nmid \alpha$. Then we have by ‘provability of unprovability’ $\vdash \neg(\sigma \rightsquigarrow \alpha)$ and thus by (ii) $\vdash \alpha$, again contradicting the hypothesis.

For the second claim suppose that $\nmid \neg\alpha$, i.e., α is consistent, and $\nmid \alpha$. It follows that $\vdash_\alpha \neq \vdash$ and $\vdash_\alpha \neq 0$. The claim would then imply $\vdash_\alpha \in [\neg(\sigma \rightsquigarrow \alpha)]$. But, since by Lemma 19 $[\neg(\sigma \rightsquigarrow \alpha)] = \{\vdash, 0\}$, this is impossible. Now suppose $\vdash \alpha$. Then (i) would

imply $\vdash \neg(\sigma \leadsto \alpha)$. But by Lemma 19 we know that this is not true, since $[\neg(\sigma \leadsto \alpha)] = [\perp]$. Thus the second claim cannot hold. \square

We get the corollary as follows. Let α be as described in the hypothesis. Then we have by ‘provability of unprovability’ that $\vdash \neg(\sigma \leadsto \alpha)$. By the above theorem it does not hold, however, that $\alpha \vdash \neg(\sigma \leadsto \alpha)$. We may interpret this as saying that, given a consistent but unprovable formula α , then the unprovability of α can be proved in \vdash , but not from α .

Given a complete classical theory Σ . Then the logic $\mathcal{L}_\Sigma = \langle \mathcal{C}_\Sigma, F_\Sigma, \rightarrow \rangle$ has only two consequence relations: $\mathcal{C}_\Sigma = \{\vdash_\Sigma, 0\}$. For the next theorem assume Fml to be the language of propositional logic.

Theorem 15. *Let \vdash be a consistent consequence relation having a pointer σ to itself. Then \vdash is monotonic iff there exists complete classical theory Σ such that $\vdash = \vdash_\Sigma$.*

Proof. The direction from right to left is obvious.

Now suppose that \vdash is monotonic. Let α be such that $\not\vdash \alpha$. Then we have by ‘provability of unprovability’ that $\vdash \neg(\sigma \leadsto \alpha)$ and by monotonicity $\alpha \vdash \neg(\sigma \leadsto \alpha)$. Since $[\neg(\sigma \leadsto \alpha)] = \{\vdash, 0\}$, we have $\vdash \neg\alpha$. Thus \vdash is complete as a consequence relation. Put $\Sigma =: \{\alpha \mid \vdash \alpha\}$. We now observe that Σ has the following properties:

- (i) $\alpha \wedge \beta \in \Sigma$ iff $\alpha \in \Sigma$ and $\beta \in \Sigma$;
- (ii) $\alpha \vee \beta \in \Sigma$ iff $\alpha \in \Sigma$ or $\beta \in \Sigma$;
- (iii) $\neg\alpha \in \Sigma$ iff $\alpha \notin \Sigma$.

It is then a fact of classical logic that Σ is classically consistent and complete.

(i) follows from the conditions we require a class of consequence relations to satisfy. We get (ii) as follows. $\vdash \alpha \vee \beta$ means $\vdash \neg(\neg\alpha \wedge \neg\beta)$. By completeness of \vdash we have that $\not\vdash (\neg\alpha \wedge \neg\beta)$. It follows that $\not\vdash \neg\alpha$ or $\not\vdash \neg\beta$ and thus again by completeness of \vdash , $\vdash \alpha$ or $\vdash \beta$. (iii) expresses completeness of \vdash .

From the above it follows that $\alpha \vdash \beta$ iff $\vdash \neg\alpha$ or $\vdash \beta$, i.e., $\neg\alpha \in \Sigma$ or $\beta \in \Sigma$ or equivalently $(\alpha \rightarrow \beta) \in \Sigma$. But this means that \rightarrow is internalising. \square

Remark. We are tempted to make some intuitive remarks concerning ‘classical’ truth and its relationship with our notion *TRUE*, i.e., self-referential truth of our metalanguage *ML*. In a sense that we would like to make precise, the ‘ordinary’ notion of truth underlying the correspondence theory of truth as expressing ‘correspondence or conformity to reality’ is a special case of our notion *TRUE*. Think of a classical model in the sense of classical model theory. In such a model every formula of a certain language is either true or false. Or consider our traditional thinking about reality, especially physical reality. We are used to assuming that every sentence pertaining to a certain physical system is either true or false. This is the attitude of ‘naïve realism’ of classical physics. So given such a system, let Σ be the set of sentences that are true of the system. Thus, for any sentence α pertaining to the system we have $\alpha \in \Sigma$ or $\neg\alpha \in \Sigma$. Now, the complete theory Σ gives rise to the self-referentially complete consequence relation \vdash_Σ . We thus have two notions of truth. On the one hand, we have our metalanguage *ML* and the notion *TRUE* φ for $\varphi \in ML$. On the other hand, we have our ‘ordinary’ notion of truth

reflecting our intuition of ‘correspondence to reality’. The first notion of truth pertains to talking about consequence, the second notion of truth pertains to talking about ‘reality’. What’s the relationship between these notions of truth? The answer is that they coincide in the following sense. Given any sentence of *ML*, say $CON\alpha$ or $PROV\alpha$. Then we see that $TRUE \varphi$ iff $\overline{\varphi} \in \Sigma$, where the latter formula is obtained from φ by deleting all metasymbols such as *PROV* and *CON*. Thus $TRUE CON\varphi$ for instances reduces to the statement that φ is true. $TRUE (PROV\alpha \leftrightarrow EQUIV(\alpha, \neg PROV\perp))$ reduces to the statement that $\alpha \leftrightarrow (\alpha \leftrightarrow \top)$ is true. In this sense we may regard sentences making true statements about ‘reality’ as making true statements about some sort of consequence and we may view our notion *TRUE* of self-referential truth as a generalisation of the notion of truth underlying ‘naive realism’. Here are a few examples of self-referentially true metaformulas.

We have *TRUE* φ for the following metastatements.

- $\varphi = \neg EQUIV(\alpha, \neg PROV\alpha)$.
- $\varphi = CON\alpha \rightarrow \neg DER(\alpha, \neg PROV\alpha)$.
- $\varphi = PROV\alpha \leftrightarrow EQUIV(\alpha, \neg PROV\perp)$.

By self-referential completeness we have in all three cases $\vdash \varphi'$. Note that for the first of the examples this means that the consequence relation ‘knows’ that it has no Gödel fixed points (Theorem 14).

8.2. Connection with the modal system *D*

The last subsection gives rise to the consideration of a certain modal system known in modal logic as the system *D*. More precisely, we shall be interested only in the letterless (deictic) fragment of *D*. This means that we are only concerned with those theorems of *D* containing no propositional symbols other than \top and \perp . Such formulas are also called modal sentences.

The system *D* is obtained from the modal system *K* by adding the axiom $\Box p \rightarrow \neg \Box \neg p$. Having the provability interpretation of the box in mind, we may view the axiom as just stating consistency.

Note that we are only interested in the letterless form of the above axiom and, more generally, in all letterless formulas which are theorems of the system *D*. Call this fragment *MC*.

We also might have chosen the following way of introducing *MC*. We could have confined ourselves to the letterless fragment of the language and could have stipulated the following.

- (i) All letterless substitutional instances of classical tautologies are theorems of *MC*.
- (ii) The rules of inference for *MC* are necessitation and modus ponens.

D is a conservative extension of the system *MC* thus introduced.

Lemma 20. *For any modal sentence A we have*

- (i) $\vdash_{MC} A$ or $\vdash_{MC} \neg A$.
- (ii) $\vdash_{MC} \Box A$ iff $\vdash_{MC} A$.

Proof. The only not completely obvious case is the case $A = \Box B$. Assume that $\text{not } \vdash_{MC} A$. We need to prove that $\vdash_{MC} \neg A$. We have that $\text{not } \vdash_{MC} B$, since otherwise necessitation would give us $\vdash_{MC} A$, contrary to the assumption. By the induction hypothesis we have $\vdash_{MC} \neg B$ and by necessitation $\vdash_{MC} \Box \neg B$. Now, since $\vdash_{MC} \Box \neg B \rightarrow \neg \Box B$, we get by modus ponens $\vdash_{MC} \neg \Box B$, which means $\vdash_{MC} \neg A$. (ii) Assume $\vdash_{MC} \Box A$ and $\text{not } \vdash_{MC} A$. By (i) we then have that $\vdash_{MC} \neg A$ and thus by necessitation $\vdash_{MC} \Box \neg A$. Since $\vdash_{MC} \Box \neg A \rightarrow \neg \Box A$, modus ponens gives us $\vdash_{MC} \neg \Box A$, which contradicts the consistency of the system D . \square

As an immediate corollary of the above theorem we get, using necessitation, that $\text{not } \vdash_{MC} A$ implies $\vdash_{MC} \neg \Box A$. This can be viewed as the ‘modal version’ of ‘provability of unprovability’. We could of course have established the above result semantically using the fact that the system D is complete for the class of Kripke frames such that for any possible world there exists a world accessible from it.

For the next theorem we use the terms ‘self-referentially correct’ and ‘self-referentially complete’ as defined by Smullyan in ‘Forever Undecided’ [24]. Recall that the main clause in the definition of self-referential truth is

$$TRUE \Box A \quad \text{iff} \quad \vdash_{MC} A.$$

Theorem 16. *MC is self-referentially correct and complete. In particular it can prove its own consistency.*

Proof. For self-referential correctness it suffices to prove that the axioms are true. Let us check the axiom $\Box \top \rightarrow \neg \Box \perp$. Since $\vdash_{MC} \top$, $\Box \top$ is true and we need to show that $\neg \Box \perp$ is true. But this is the case iff the system is consistent, which is well known from modal logic.

For self-referential completeness we need to prove that for any A the truth of A implies $\vdash A$. Again, the only cases not completely obvious are the cases $A = \Box B$ and $A = \neg B$. Suppose $A = \Box B$ is true. This means $\vdash B$. By necessitation we have $\vdash \Box B$ and thus $\vdash A$. Suppose $A = \neg B$ is true. This says that B is not true and thus by the induction hypothesis $\text{not } \vdash B$. By Lemma 20 this means $\vdash \neg B$ and thus $\vdash A$. \square

Since the system D is consistent, we have $TRUE \neg \Box \perp$ and thus by self-referential completeness $\vdash_{MC} \neg \Box \perp$. It is in this sense that we can say that D can prove its own consistency. Recall that the famous modal system G for instance does not have this property, because by Solovay’s completeness theorem this would contradict Gödel’s incompleteness theorem (see [1]).

Let us now describe the connection between the considerations of the last subsection and the system MC . Consider the fragment MLP of the metalanguage ML consisting of those formulas in which only $PROV$ occurs. More precisely, it is the following (meta-) language. It is the smallest set such that

- If α is a wff of the object language, then $PROV\alpha \in MLP$.
- If $\varphi \in MLP$, then $PROV\varphi \in MLP$.
- If $\varphi, \psi \in MLP$, so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$.

Let \sim be a consequence relation, α a wff of the object language. We say that α has truth value \top with respect to \sim , if $\sim \alpha$, otherwise we say that α has truth value \perp with respect to \sim . Now, given a formula $\varphi \in MLP$ and a consequence relation \sim having a pointer to itself. Then we define φ^d to be the modal sentence resulting from φ by replacing every occurrence of *PROV* by the modal operator \Box and all occurrences of wffs of the object language by their truth values with respect to \sim . Then we have the

Theorem 17. *Let \sim have a pointer to itself, let $\varphi \in MLP$. Then the following statements are equivalent.*

- (i) *TRUE* φ .
- (ii) $\sim \varphi'$.
- (iii) $\vdash_{MC} \varphi^d$.
- (iv) *TRUE* φ^d .

Remark. Note that in (i) *TRUE* is self-referential truth in the sense of Theorem 13, whereas in (iv) *TRUE* means self-referential truth in the modal sense.

Proof. The equivalence between (i) and (ii) is given by self-referential completeness of \sim , the equivalence of (iii) and (iv) is given by self-referential completeness of the modal system *MC*.

We need to prove that (i) and (iii) are equivalent. Case $\varphi = PROV\alpha$, where α is a wff of the object language. Assume *TRUE* φ . This means $\sim \alpha$ and thus $\varphi^d = \Box\top$ and, clearly, $\vdash_{MC} \varphi^d$. For the other direction note that either $\varphi^d = \Box\top$ or $\varphi^d = \Box\perp$. Since $\vdash_{MC} \varphi^d$, we have $\varphi^d = \Box\top$. It follows that the truth value of α must be \top and thus $\sim \alpha$. But this means *TRUE* $PROV\alpha$.

Case $\varphi = PROV\psi$. Assume *TRUE* φ . This means $\sim \psi'$. We have $\varphi^d = \Box\psi^d$ and by the induction hypothesis $\vdash_{MC} \psi^d$. Hence $\vdash_{MC} \Box\psi^d$, which says $\vdash_{MC} \varphi^d$. For the other direction let $\vdash_{MC} \varphi^d$, i.e., $\vdash_{MC} \Box\psi^d$. By Lemma 20 this is equivalent to $\vdash_{MC} \psi^d$. The induction hypothesis yields $\sim \psi'$. But this says *TRUE* φ .

The other cases are straightforward. \square

9. Conclusion and outlook

The starting point for the considerations presented in this paper is the observation that there exist close connections between two different fields of logical research having been developed independently and having grown from completely different sources.

The first type of research has its origin in the field of *Artificial Intelligence*, especially in investigations on non-monotonic reasoning and belief revision. The second type of research is what has become known as *Quantum Logic*, a discipline having its roots in investigations on the foundations of *Quantum Mechanics*. It is in this context that the mathematical concept of a *Hilbert space* was realised to be of logical significance.

It turns out that the ideas and concepts developed in the first field of research can be used to cast light on the nature of the connection between Hilbert space and logic, which is an issue of vital importance for the interpretation of quantum mechanics. It turns

out that Hilbert space constitutes a beautiful vehicle for illustrating ideas and concepts developed in *AI* research. In particular, there is overwhelming evidence in the light of that research that from a logical point of view the mathematical structure of a Hilbert space should be regarded as representing a system of (non-monotonic) consequence relations interconnected by formulas in a natural way. This observation can be generalised in a meaningful way and has led us to the concept of a *consequence revision system*. This concept is of logical interest beyond the field of quantum logic.

One of the newly emerging and promising fields of logical research is that of combining logical systems. A general methodology for combining logics was developed in [6] and [7]. In a subsequent paper we shall investigate the way Hilbert space logics combine via tensor product formation.

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